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**Memorandum No. 1752**

# The D-Boussinesq equation: Hamiltonian and symplectic structures; Noether and inverse Noether operators

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February, 2005

ISSN 0169-2690

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# THE D-BOUSSINESQ EQUATION: HAMILTONIAN AND SYMPLECTIC STRUCTURES; NOETHER AND INVERSE NOETHER OPERATORS

A. VERBOVETSKY, P. KERSTEN*, AND I. KRASIL′SHCHIK

Abstract. Using new methods of analysis of integrable systems,based on a general geometric approach to nonlinear PDE,we discuss the Dispersionless Boussinesq Equation, which is equivalent to the Benney-Lax equation,being a system of equations of hydrodynamical type.
The results include: a description of local and nonlocal Hamiltonian and symplectic structures, hierarchies of symmetries, hierarchies of conservation laws, recursion operators for symmetries and generating functions of conservation laws. Highly interesting are the appearences of the Noether and Inverse Noether operators ,leading to multiple infinite hierarchies of these operators as well as recursion operators.

## Introduction

The equation (or system of equations) under study is the Dispersionless Boussinesq Equation [11], [12], a quasi linear system of first order in three dependent $(w, u, v)$ and two independent $(x, t)$ variables , being described as

$$w_t = u_x, \tag{1}$$

$$u_t = ww_x + v_x, \tag{2}$$

$$v_t = -uw_x - 3wu_x, \tag{3}$$

The aim of our paper is twofold: (1) to represent the known results in a more convenient form (at least, from our point of view); (2) to demonstrate the efficiency of new methods of analysis of integrable systems described in [6, 7] and based on a general geometric approach to nonlinear PDE [2, 9]. Actually, description of *these methods and their highly algorithmical nature in applications* is the main goal of the paper. For traditional approach to Hamiltonian formalism in integrable systems we refer the reader to [3, 10, 14, 16]

This paper is organized as follows.
In Section 1, we present the essential definitions and results needed for applications paying main attention to the computational aspects rather than to theoretical ones. All the proofs can be found in [2, 6, 7, 9].
In Section 2, the results for the D-Boussinesq Equation are described.
Finally, in the last section we briefly discuss the results and perspectives.

## 1. Description of the computational scheme

Here we deal with evolution systems $\mathcal{E}$ of the form

$$v_t = F(y, t, v_1, \ldots, v_k), \tag{4}$$

2000 *Mathematics Subject Classification.* 37K05, 35Q53.
2003 *PACS*: 02.30.Ik; 11.30.-j.
*Key words and phrases.* symmetry, conservation law, Hamiltonian structure, symplectic structure, Noether operator.
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where both the unknown variable $v = (v^1, \dots, v^m)$ and the right-hand side $F = (F^1, \dots, F^m)$ are vector-functions and $v_i = \partial^i v / \partial y^i$, $y$ and $t$ being the independent variables.
Two basic operators related to (4),

$$D_y = \frac{\partial}{\partial y} + \sum_{i,j} v^j_{i+1} \frac{\partial}{\partial v^j_i}, \quad D_t = \frac{\partial}{\partial t} + \sum_{i,j} D^i_y(F^j) \frac{\partial}{\partial v^j_i},$$

are called the *total derivatives*.

*Remark* 1. Note that the above expressions for total derivatives contain infinite number of terms. To make the action of these operators (as well as of similar operators introduced below) well defined, we introduce the space $\mathcal{F}(\mathcal{E})$ of functions smoothly depending on $y$, $t$ and a *finite number* of variables $v^j_i$, and assume $D_y$ and $D_t$ to act in this space. Similarly, we shall consider the spaces $\mathcal{F}^m(\mathcal{E})$ of vector-functions of length $m$ that depend on $y$, $t$ and $v^j_i$ in the same way.

1.1. **Symmetries.** A *symmetry* of equation (4) is a vector field

$$S = \sum_{i,j} S^j_i \frac{\partial}{\partial v^j_i}, \quad S^j_i \in \mathcal{F}(\mathcal{E}),$$

such that

$$[S, D_y] = [S, D_t] = 0.$$

Any symmetry is of the form

$$\partial_f = \sum_{i,j} D^i_y(f^j) \frac{\partial}{\partial v^j_i}, \tag{5}$$

where the vector-function $f = (f^1, \dots, f^m) \in \mathcal{F}^m(\mathcal{E})$ satisfies the system of equations

$$D_t(f^l) = \sum_{i,j} \frac{\partial F^l}{\partial v^j_i} D^i_y(f^j), \quad l = 1, \dots, m. \tag{6}$$

The operator at the right-hand side of (6) is called the *linearization* of $F$ and is denoted by $\ell_F$. Thus, equation (6) acquires the form

$$D_t(f) = \ell_F(f). \tag{7}$$

There exists a one-to-one correspondence between symmetries (5) and the corresponding functions $f \in \mathcal{F}^m(\mathcal{E})$, hence we shall identify symmetries with such functions and use the term ‘symmetry’ for any function that satisfy (7).

1.2. **Conservation laws and generating functions.** A *conservation law* of system (4) is a pair $\Omega = (Y, T)$, $Y$, $T \in \mathcal{F}(\mathcal{E})$, such that

$$D_t(Y) = D_y(T). \tag{8}$$

The function $Y$ is called the *density* of $\Omega$. A conservation law is called *trivial* if $Y = D_y(P)$, $T = D_t(P)$ for some function $P \in \mathcal{F}(\mathcal{E})$.

To any conservation law there corresponds its *generating function* defined by

$$g_\Omega = \frac{\delta Y}{\delta v} = \left( \frac{\delta Y}{\delta v^1}, \dots, \frac{\delta Y}{\delta v^m} \right),$$

where

$$\frac{\delta}{\delta v^j} = \sum_{i \geq 0} (-D_y)^i \circ \frac{\partial}{\partial v^j_i}$$

is the *variational derivative* with respect to $v^j$. Generating functions of conservation laws satisfy the system of equations

$$D_t(g) = -\ell^*_F(g), \tag{9}$$

or

$$D_t(g^l) = -\sum_{i,j}(-D_y)^i\left(\frac{\partial F^j}{\partial v_i^l}g^j\right), \quad l = 1,\dots,m, \tag{10}$$

where $\ell_F^*$ is *adjoint* to the operator $\ell_F$.

Any conservation law is uniquely determined by its generating function and, in particular, $\Omega$ is trivial if and only if $g_\Omega = 0$. Stress that equation (10) may possess solutions that do not correspond to any conservation law of (4).

*Remark* 2. Generating functions are also called *cosymmetries* [1] or *conserved covariants* [4].

1.3. **Nonlocal variables.** Let us introduce a set of variables $w^1,\dots,w^j,\dots$ satisfying the equations

$$w_y^j = A^j(y,t,\dots,v_i^\alpha,\dots,w^\beta,\dots), \quad w_t^j = B^j(y,t,\dots,v_i^\alpha,\dots,w^\beta,\dots), \tag{11}$$

that are compatible modulo equation (4), where $A^j$, $B^j$ are some smooth functions depending on a finite number of arguments. Consider the operators

$$\widetilde{D}_y = D_y + \sum_j A^j\frac{\partial}{\partial w^j}, \quad \widetilde{D}_t = D_t + \sum_j B^j\frac{\partial}{\partial w^j}.$$

Due to the compatibility conditions, one has

$$[\widetilde{D}_y, \widetilde{D}_t] = 0 \tag{12}$$

modulo (4). The variables $w^j$ are called *nonlocal.*

Using the operators $\widetilde{D}_y$, $\widetilde{D}_t$ instead of $D_y$ and $D_t$ in formulas (6), (8), and (10), we can introduce the notions of *nonlocal symmetries*, *nonlocal conservation laws*, and *nonlocal generating functions* depending on the new variables $w^j$. We shall denote the spaces of such symmetries and generating functions by $\mathbf{sym}(\mathcal{E})$ and $\mathbf{gf}(\mathcal{E})$, respectively.

*Remark* 3. An invariant geometric way to introduce nonlocal variables is based on the notion of *covering*, see [2, 7, 8, 9].

1.4. **The $\ell$- and $\ell^*$-extensions.** There are two canonical ways to extend the initial system (4). The first one is related to the operator $\ell_F$ and is called the *$\ell$-extension.* Namely, let us introduce the nonlocal variables $\omega_i^j$ (we shall also denote $\omega_0^j$ by $\omega^j$), $j = 1,\dots,m$, $i = 0, 1, \dots$, satisfying the relations

$$(\omega_i^j)_y = \omega_{i+1}^j, \quad (\omega_i^j)_t = \widetilde{D}_y^i\left(\sum_{s,l}\frac{\partial F^j}{\partial v_s^l}\omega_s^l\right).$$

Clearly, these equations are consistent modulo (4) and are the consequences of the following ones

$$\omega_t^j = \sum_{i,l}\frac{\partial F^j}{\partial v_i^l}\omega_i^l. \tag{13}$$

In a similar way we construct the *$\ell^*$-extension*: the nonlocal variables are $\pi_i^j$ ($\pi_0^j$ will also be denoted by $\pi^j$) and the defining relations are

$$(\pi_i^j)_y = \pi_{i+1}^j, \quad (\pi_i^j)_t = -\widetilde{D}_y^i\left(\sum_{s,l}(-\widetilde{D}_y)^s\left(\frac{\partial F^l}{\partial v_s^j}\pi^l\right)\right),$$

that reduce to the equations

$$\pi_t^j = -\sum_{s,l}(-\widetilde{D}_y)^s\left(\frac{\partial F^l}{\partial v_s^j}\pi^l\right) \tag{14}$$

and their differential consequences.

*Remark* 4. The parities of the variables $\omega^j$ and $\pi^j$ are opposite to that of $v^j$: if $v^j$ is *even*, then $\omega^j$ and $\pi^j$ are *odd* and vice versa.

If the initial equation $\mathcal{E}$ was extended by nonlocal variables $w^j$, we can associate to these variables, in a canonical way, the corresponding $\omega$'s and $p$'s whose 'behavior' is governed by linearization or, respectively, adjoint linearization of equations (11) in the corresponding nonlocal setting.

*Associating operators to functions on the $\ell$- and $\ell^*$-extensions.* Let $\mathcal{F}^m(\mathcal{E})$ be the space of vector-valued functions of length $m$ (see Remark 1). Consider the case when $\mathcal{E}$ is not extended by nonlocal variables first. Let $a = (a_1, \dots, a_m)$, $a_i = \sum_{jl} a_l^{ij}\omega_l^j$, $a_l^{ij} \in \mathcal{F}(\mathcal{E})$, be a linear in $\omega$ vector-function. Then we put into correspondence to this function a differential operator $\Delta_a = \|\Delta_a^{ij}\| \colon \mathcal{F}^m(\mathcal{E}) \to \mathcal{F}^m(\mathcal{E})$, where

$$\Delta_a^{ij} = \sum_l a_l^{ij} D_y^l, \qquad i, j = 1, \dots, m.$$

If $\mathcal{F}(\mathcal{E})$ contains nonlocal variables, the situation becomes more complicated. We shall consider here the simplest case when the functions $A^j$ in (11) are independent of $\omega^\beta$. Let $\bar{\omega}^\beta$ be the variable in the $\ell$-extension associated to the nonlocal variable $w^\beta$ and $b = (b^1, \dots, b^m)$, $b^i = \sum_\beta b^{i\beta}\bar{\omega}^\beta$, be a linear in $\bar{\omega}$ vector-function. Then the corresponding operator $\Delta_b = \|\Delta_b^{ij}\| \colon \mathcal{F}^m(\mathcal{E}) \to \mathcal{F}^m(\mathcal{E})$ is of the form

$$\Delta_b^{ij} = \sum_\alpha b^{i\alpha} D_y^{-1} \circ \sum_l \frac{\partial A^\alpha}{\partial v_l^j} D_y^l. \tag{15}$$

For the $\ell^*$-extension the construction is completely similar.

Below we shall use the notation $\mathcal{L}^m(\ell_\mathcal{E})$ and $\mathcal{L}^m(\ell_\mathcal{E}^*)$ for the spaces of vector-functions linear in $\omega$, $\bar{\omega}$ and $p$, $\bar{p}$, respectively.

1.5. **Recursion operators for symmetries.** Let $R \in \mathcal{L}^m(\ell_\mathcal{E})$ be a function that satisfies the equation

$$\widetilde{D}_t(R) = \widetilde{\ell}_F(R).$$

Then the corresponding operator $\Delta_R$ maps $\mathbf{sym}(\mathcal{E})$ to $\mathbf{sym}(\mathcal{E})$ and thus is a recursion operator for (nonlocal) symmetries of $\mathcal{E}$.

*Remark* 5. Here and below $\widetilde{\ell}_F$ denotes the linearization operator with the total derivative $D_y$ replaced by its counterpart $\widetilde{D}_y$ for the $\ell$- or $\ell^*$-covering, and $\widetilde{\ell}_F^*$ stands for the adjoint of $\widetilde{\ell}_F$.

1.6. **Recursion operators for generating functions.** Let $L \in \mathcal{L}^m(\ell_\mathcal{E}^*)$ be a function that satisfies the equation

$$\widetilde{D}_t(L) = -\widetilde{\ell}_F^*(L).$$

Then the corresponding operator $\Delta_L$ maps $\mathbf{gf}(\mathcal{E})$ to $\mathbf{gf}(\mathcal{E})$ and thus is a recursion operator for (nonlocal) generating functions of $\mathcal{E}$ (or *adjoint recursion operator* [1]).

1.7. **Hamiltonian structures.** Let $K \in \mathcal{L}^m(\ell_\mathcal{E}^*)$ be a function that satisfies the equation

$$\widetilde{D}_t(K) = \widetilde{\ell}_F(K).$$

Then the corresponding operator $\Delta_K$ maps $\mathbf{gf}(\mathcal{E})$ to $\mathbf{sym}(\mathcal{E})$. We call such maps *pre-Hamiltonian structures* (they are also known as *Noether operators* [4]). In order $\Delta_K$ to be a true *Hamiltonian structure*, it has to satisfy two conditions: skew-symmetry ($\Delta_K^* = -\Delta_K$) and the Jacobi identity for the corresponding Poisson bracket (that amounts to $[\![\Delta_K, \Delta_K]\!] = 0$, where $[\![\cdot,\cdot]\!]$ is the *variational Schouten bracket*, see [5, 6]). Both these conditions are easily checked in terms of the function $K$.

Namely, if $K = \left\| \sum_{jl} a_l^{ij} \pi_l^j \right\|$ then we consider the function $W_K = \sum_{ijl} a_l^{ij} \pi_l^j \pi^i$ and in terms of $W_K$ the first condition reads

$$\sum_i \frac{\delta W_K}{\delta \pi^i} \pi^i = -2W_K, \tag{16}$$

while the second one is

$$\left( \frac{\delta}{\delta v}, \frac{\delta}{\delta p} \right) \sum_i \left( \frac{\delta W_K}{\delta v^i} \frac{\delta W_K}{\delta \pi^i} \right) = 0, \tag{17}$$

$(\delta/\delta v, \delta/\delta p) = (\delta/\delta v^1, \dots, \delta/\delta v^m, \delta/\delta \pi^1, \dots, \delta/\delta \pi^m)$. Note also that the compatibility condition for two Hamiltonian structures $K$ and $K'$ amounts to

$$\left( \frac{\delta}{\delta v}, \frac{\delta}{\delta p} \right) \sum_i \left( \frac{\delta W_K}{\delta v^i} \frac{\delta W_{K'}}{\delta \pi^i} + \frac{\delta W_{K'}}{\delta v^i} \frac{\delta W_K}{\delta \pi^i} \right) = 0. \tag{18}$$

The equation $\mathcal{E}$ itself is in the Hamiltonian form if it possesses a Hamiltonian structure $K$ and may be presented as

$$v_t = \Delta_K \frac{\delta Y}{\delta v} \tag{19}$$

for some function $Y$.

1.8. **Symplectic structures.** Let $J \in \mathcal{L}^m(\ell_{\mathcal{E}})$ be a function that satisfies the equation

$$\widetilde{D}_t(J) = -\widetilde{\ell}_F^*(J).$$

Then the corresponding operator $\Delta_J$, which maps $\mathbf{sym}(\mathcal{E})$ to $\mathbf{gf}(\mathcal{E})$, is called an *Inverse Noether operator* [4] for $\mathcal{E}$. An operator $\Delta_J \colon \mathbf{sym}(\mathcal{E}) \to \mathbf{gf}(\mathcal{E})$, not necessary being an iverse Noether operator, is called *symplectic* (or a *symplectic structure*) cf. e.g. [4, 13], if it enjoys the following properties. Let $J = \left\| \sum_{jl} b_l^{ij} \omega_l^j \right\|$. Similar to Subsection 1.7, we consider the function $W_J = \sum_{ijl} b_l^{ij} \omega_l^j \omega^i$ and impose the conditions

$$\sum_i \frac{\delta W_J}{\delta \omega^i} \omega^i = -2W_J, \tag{20}$$

i.e., the operator $\Delta_J$ is skew-adjoint, and

$$\left( \frac{\delta}{\delta v}, \frac{\delta}{\delta \omega} \right) \sum_i \frac{\delta W_J}{\delta v^i} \omega^i = 0 \tag{21}$$

that means that the 'form' $W_J$ s is closed. Thus, in our context the term 'symplectic structure' means the same as in classical mechanics, cf. [13].

1.9. **Canonical representation.** As it will be seen below, all the operators constructed in our study are presented in the form

$$\sum_{\alpha \ge 0} c_{ij}^\alpha D_y^\alpha + \sum_\beta d_j^\beta D_y^{-1} \circ e_i^\beta,$$

where $\left\| c_{ij}^\alpha \right\|$ is an $m \times m$-matrix, $\left\| d_j^\beta \right\|$ is an $m \times l$-matrix, and $\left\| e_i^\beta \right\|$ is an $l \times m$-matrix for some $l > 0$ (matrix-valued functions, to be more precise). In the table it is shown how the matrices $d$ and $e$ look for different types of operators.

| Type of operator | Lines of matrix $d$ | Columns of matrix $e$ |
|---|---|---|
| Recursions for symmetries | Symmetry | Generating function |
| Recursions for generating funct. | Generating function | Symmetry |
| Hamiltonian structures | Symmetry | Symmetry |
| Symplectic structures | Generating function | Generating function |

## 2. Main results for the D-Boussinesq Equation

Here we apply the theory described above in Section I to equation (1)

$$\begin{aligned} w_t &= u_x, \\ u_t &= ww_x + v_x, \\ v_t &= -uw_x - 3wu_x, \end{aligned} \tag{22}$$

where we shall use the notation

$$w_t = \frac{\partial w}{\partial t}, w_x = w_1 = \frac{\partial w}{\partial x}, w_k = \frac{\partial^k w}{\partial x^k} \ (k = 1, \ldots), u_t = \frac{\partial u}{\partial t}, \ldots.$$

Remark:**Gradings**
We assign the following gradings or degrees $[\cdot]$ to the variables of our equation:

$$[w] = 2, \quad [u] = 3, \quad [v] = 4 \quad [t] = -3, \quad [x] = -1.$$

So, we have

$$[u_3] = 3 + 3 = 6$$

With these gradings, equations (1) become homogeneous (of grading $4, 5, 6$) and all constructions below can be considered to be homogeneous as well.

2.1. **Nonlocal functions.** Here we extend the equation $\mathcal{E}$ by a group of nonlocal variables. These nonlocal variables result from conservation laws

$$D_t(F) = D_x(G),$$

where $F, G$ are dependent on $x, t, w, u, v, \ldots$,while $F$ is called the density of the conservation law.
Formally we denote the associated nonlocal variable by

$$p = D_x^{-1}(F)$$

We present here the $x$-and $t$-derivatives of these nonlocal variables , together discarding ambiguity.

2.1.1. *Group 1.* This group includes the variables $p_1$, $p_2$, $p_3$, $p_5$, $p_6$, $p_7$, $p_9$, $p_{10}$, $p_{11}$, $p_{13}$, $p_{14}$, $p_{15}$, and $p_{17}$ defined by

$$\begin{aligned}
(p_1)_x &= w, & (p_1)_t &= u, \\
(p_2)_x &= u, & (p_2)_t &= (2v + w^2)/2, \\
(p_3)_x &= v + w^2, & (p_3)_t &= -uw, \\
(p_5)_x &= (u^2 + 2vw + 2w^3)/2, & (p_5)_t &= uv, \\
(p_6)_x &= u(2v + w^2), & (p_6)_t &= (-8u^2w + 4v^2 + 4vw^2 + w^4)/4, \\
(p_7)_x &= (-6u^2w + 6v^2 + 12vw^2 + 7w^4)/7, & (p_7)_t &= (2u(-u^2 - 6vw - 4w^3))/7, \\
(p_9)_x &= (10u^2v + 10v^2w + 20vw^3 + 11w^5)/11, & (p_9)_t &= (5u(-2u^2w + 2v^2 - w^4))/11,
\end{aligned}$$

$$
\begin{aligned}
(p_{10})_x &= (u(-8u^2w + 12v^2 + 12vw^2 + 3w^4))/3,\\
(p_{10})_t &= (-4u^4 - 48u^2vw - 24u^2w^3 + 8v^3 + 12v^2w^2 + 6vw^4 + w^6)/6,\\
(p_{11})_x &= (-u^4 - 12u^2vw - 8u^2w^3 + 4v^3 + 12v^2w^2 + 14vw^4 + 6w^6)/6,\\
(p_{11})_t &= (u(-2u^2v + 2u^2w^2 - 6v^2w - 8vw^3 - 3w^5))/3,\\
(p_{13})_x &= (-15u^4w + 30u^2v^2 - 15u^2w^4 + 20v^3w + 60v^2w^3 + 66vw^5 + 26w^7)/26,\\
(p_{13})_t &= (u(-3u^4 - 60u^2vw - 20u^2w^3 + 20v^3 - 30vw^4 - 16w^6))/26,\\
(p_{14})_x &= (u(-4u^4 - 80u^2vw - 40u^2w^3 + 40v^3 + 60v^2w^2 + 30vw^4 + 5w^6))/5,\\
(p_{14})_t &= (-32u^4v + 48u^4w^2 - 192u^2v^2w - 192u^2vw^3 - 48u^2w^5 + 16v^4 + 32v^3w^2\\
&\quad + 24v^2w^4 + 8vw^6 + w^8)/8,\\
(p_{15})_x &= (-28u^4v + 28u^4w^2 - 168u^2v^2w - 224u^2vw^3 - 84u^2w^5 + 28v^4 + 112v^3w^2\\
&\quad + 196v^2w^4 + 168vw^6 + 57w^8)/57,\\
(p_{15})_t &= (4u(7u^4w - 14u^2v^2 + 28u^2vw^2 + 21u^2w^4 - 28v^3w - 56v^2w^3 - 42vw^5 - 12w^7))/57,\\
(p_{17})_x &= (-6u^6 - 180u^4vw - 60u^4w^3 + 120u^2v^3 - 180u^2vw^4 - 96u^2w^6 + 60v^4w + 240v^3w^3\\
&\quad + 396v^2w^5 + 312vw^7 + 97w^9)/97,\\
(p_{17})_t &= (3u(-12u^4v + 24u^4w^2 - 120u^2v^2w - 80u^2vw^3 - 4u^2w^5 + 20v^4 - 60v^2w^4\\
&\quad - 64vw^6 - 21w^8))/97,
\end{aligned}
$$

The indices $*$ of the $p_*$ refer to the associated gradings of $p_*$,

$$[p_1] = 1, [p_2] = 2, [p_3] = 3, \ldots.$$

2.1.2. *Group 2.* This group includes the variables $q_0$, $q_4$, $q_8, \overline{q}_8$, $q_{12}, \overline{q}_{12}$ defined by nonlocal conservation laws

$$\widetilde{D_t}(F) = \widetilde{D_x}(G),$$

where now $F, G$ are dependent on $x, t, w, u, v, \ldots, p_1, p_2, \ldots$,while $F$ is again the density of the (nonlocal) conservation law.
Formally we denote the associated nonlocal variable by

$$q = \widetilde{D_x}^{-1}(F).$$

We present here the $x$-and $t$-derivatives of these nonlocal variables , together discarding ambiguity.
This gives rise to

$$
\begin{aligned}
(q_0)_x &= p_1,\\
(q_0)_t &= p_2,\\
(q_4)_x &= p_3w - 3p_5,\\
(q_4)_t &= p_3u - 2p_6,\\
(q_8)_x &= (4(700p_5v + 700p_5w^2 + 320p_6u + 315p_7w - 121p_9))/55,\\
(q_8)_t &= (4(-140p_5uw + 64p_6v + 32p_6w^2 + 63p_7u))/11,\\
(\overline{q}_8)_x &= (144p_5v + 144p_5w^2 + 63p_6u + 56p_7w)/144,\\
(\overline{q}_8)_t &= (33p_{10} - 576p_5uw + 252p_6v + 126p_6w^2 + 224p_7u)/576,\\
(q_{12})_x &= (192p_{10}u + 390p_{11}w - 182p_{13} - 147p_7u^2 - 294p_7vw - 294p_7w^3 + 726p_9v + 726p_9w^2)/33,\\
(q_{12})_t &= (2(32p_{10}v + 16p_{10}w^2 + 65p_{11}u - 49p_7uv - 121p_9uw))/11,\\
(\overline{q}_{12})_x &= (-33p_{10}u - 60p_{11}w + 28p_7u^2 + 56p_7vw + 56p_7w^3 - 132p_9v - 132p_9w^2)/56,
\end{aligned}
$$

$(\overline{q}_{12})_t = (-66p_{10}v - 33p_{10}w^2 - 120p_{11}u - 10p_{14} + 112p_7uv + 264p_9uw)/112,$

The indices $*$ of the $q_*$ refer to the associated gradings of $q_*$,

$$[q_0] = 0, [q_4] = 4, [q_8] = [\overline{q}_8] = 8, [q_{12}] = [\overline{q}_{12}] = 12.$$

2.2. **Symmetries.** In order to construct symmetries of the D-Boussinesq equation

$$S = S^w\partial_w + S^u\partial_u + S^v\partial_v$$

we have to solve equation (6), which in our case is of the form

$$\begin{aligned}
&\widetilde{D}_t(S^w) = \widetilde{D}_x(S^u),\\
&\widetilde{D}_t(S^u) = w\widetilde{D}_x(S^w) + w_1S^w + \widetilde{D}_x(S^v),\\
&\widetilde{D}_t(S^v) = -u\widetilde{D}_x(S^w) - 3u_1S^w - 3w\widetilde{D}_x(S^u) - w_1S^u
\end{aligned} \tag{23}$$

where $\widetilde{D}_x$ and $\widetilde{D}_t$ are the total derivative operators extended to the nonlocal setting (see Subsection 2.1),including $p_*, q_*$ we found a number of solutions leading to infinite hierarchies of symmetries and which are also used to construct *nonlocal vectors* (see Subsection 2.4 below).
These symmetries are given as

2.2.1. $(x, t)$ - *independent Symmetries.*

$$\begin{aligned}
S^w(-4) &= 0,\\
S^u(-4) &= 0,\\
S^v(-4) &= 1,
\end{aligned}$$

$$\begin{aligned}
S^w(1) &= w_1,\\
S^u(1) &= u_1,\\
S^v(1) &= v_1,
\end{aligned}$$

$$\begin{aligned}
S^w(2) &= u_1,\\
S^u(2) &= w_1w + v_1,\\
S^v(2) &= -w_1u - 3u_1w,
\end{aligned}$$

$$\begin{aligned}
S^w(3) &= 2w_1w + v_1,\\
S^u(3) &= -w_1u - u_1w,\\
S^v(3) &= -3w_1w^2 - u_1u - 2v_1w,
\end{aligned}$$

$$\begin{aligned}
S^w(5) &= w_1(v + 3w^2) + u_1u + v_1w,\\
S^u(5) &= u_1v + v_1u,\\
S^v(5) &= w_1(-u^2 - 3w^3) - 4u_1uw + v_1(v - w^2),
\end{aligned}$$

$$\begin{aligned}
S^w(6) &= 2w_1uw + u_1(2v + w^2) + 2v_1u,\\
S^u(6) &= w_1(-2u^2 + 2vw + w^3) - 4u_1uw + v_1(2v + w^2),\\
S^v(6) &= w_1u(-2v - 7w^2) + u_1(-2u^2 - 6vw - 3w^3) - 6v_1uw,
\end{aligned}$$

$$S^w(7) = w_1(-3u^2 + 12vw + 14w^3) - 6u_1uw + 6v_1(v + w^2),$$

$$
\begin{aligned}
S^u(7) &= 6w_1u(-v-2w^2)+u_1(-3u^2-6vw-4w^3)-6v_1uw,\\
S^v(7) &= 6w_1w(u^2-3vw-3w^3)+6u_1u(-v+2w^2)+v_1(-3u^2-12vw-10w^3),
\end{aligned}
$$

$$
\begin{aligned}
S^w(9) &= w_1(2v^2+12vw^2+11w^4)+4u_1uv+2v_1(u^2+2vw+2w^3),\\
S^u(9) &= 2w_1u(-u^2-2w^3)+u_1(-6u^2w+2v^2-w^4)+4v_1uv,\\
S^v(9) &= 2w_1(-2u^2v-3u^2w^2-6vw^3-6w^5)+2u_1u(-u^2-8vw-2w^3)\\
&\quad+v_1(-8u^2w+2v^2-4vw^2-5w^4),
\end{aligned}
$$

$$
\begin{aligned}
S^w(10) &= 4w_1u(-2u^2+6vw+3w^3)+3u_1(-8u^2w+4v^2+4vw^2+w^4)+12v_1u(2v+w^2),\\
S^u(10) &= 3w_1(-8u^2v-12u^2w^2+4v^2w+4vw^3+w^5)+8u_1u(-u^2-6vw-3w^3)\\
&\quad+3v_1(-8u^2w+4v^2+4vw^2+w^4),\\
S^v(10) &= 3w_1u(8u^2w-4v^2-28vw^2-13w^4)\\
&\quad+3u_1(-8u^2v+20u^2w^2-12v^2w-12vw^3-3w^5)\\
&\quad+4v_1u(-2u^2-18vw-9w^3),
\end{aligned}
$$

$$
\begin{aligned}
S^w(11) &= 2w_1(-3u^2v-6u^2w^2+6v^2w+14vw^3+9w^5)+2u_1u(-u^2-6vw-4w^3)\\
&\quad+v_1(-6u^2w+6v^2+12vw^2+7w^4),\\
S^u(11) &= w_1u(4u^2w-6v^2-24vw^2-15w^4)+u_1(-6u^2v+6u^2w^2-6v^2w-8vw^3-3w^5)\\
&\quad+2v_1u(-u^2-6vw-4w^3),\\
S^v(11) &= w_1(2u^4+12u^2vw+26u^2w^3-18v^2w^2-36vw^4-21w^6)\\
&\quad+u_1u(12u^2w-6v^2+24vw^2+17w^4)+2v_1(-3u^2v+6u^2w^2-6v^2w-10vw^3-5w^5),
\end{aligned}
$$

$$
\begin{aligned}
S^w(13) &= w_1(-15u^4-60u^2w^3+20v^3+180v^2w^2+330vw^4+182w^6)\\
&\quad+30u_1u(-2u^2w+2v^2-w^4)\\
&\quad+6v_1(10u^2v+10v^2w+20vw^3+11w^5),\\
S^u(13) &= 12w_1u(-5u^2v-5u^2w^2-10vw^3-8w^5)\\
&\quad+u_1(-15u^4-180u^2vw-60u^2w^3+20v^3-30vw^4-16w^6)\\
&\quad+30v_1u(-2u^2w+2v^2-w^4),\\
S^v(13) &= 6w_1(10u^4w-10u^2v^2-30u^2vw^2+5u^2w^4-30v^2w^3-60vw^5-33w^7)\\
&\quad+12u_1u(-5u^2v+15u^2w^2-20v^2w-10vw^3+2w^5)\\
&\quad+v_1(-15u^4-240u^2vw-60u^2w^3+20v^3-60v^2w^2-150vw^4-82w^6),
\end{aligned}
$$

$$
\begin{aligned}
S^w(14) &= 2w_1u(-8u^2v-12u^2w^2+12v^2w+12vw^3+3w^5)\\
&\quad+u_1(-4u^4-48u^2vw-24u^2w^3+8v^3+12v^2w^2+6vw^4+w^6)\\
&\quad+2v_1u(-8u^2w+12v^2+12vw^2+3w^4),\\
S^u(14) &= w_1(12u^4w-24u^2v^2-72u^2vw^2-30u^2w^4+8v^3w+12v^2w^3+6vw^5+w^7)\\
&\quad+4u_1u(-4u^2v+6u^2w^2-12v^2w-12vw^3-3w^5)\\
&\quad+v_1(-4u^4-48u^2vw-24u^2w^3+8v^3+12v^2w^2+6vw^4+w^6),
\end{aligned}
$$

$$
\begin{aligned}
S^v(14) &= w_1u(4u^4 + 48u^2vw + 72u^2w^3 - 8v^3 - 84v^2w^2 - 78vw^4 - 19w^6)\\
&\quad + u_1(28u^4w - 24u^2v^2 + 120u^2vw^2 + 66u^2w^4 - 24v^3w - 36v^2w^3 - 18vw^5 - 3w^7)\\
&\quad + 2v_1u(-8u^2v + 20u^2w^2 - 36v^2w - 36vw^3 - 9w^5),
\end{aligned}
$$

$$
\begin{aligned}
S^w(15) &= w_1(14u^4w - 42u^2v^2 - 168u^2vw^2 - 105u^2w^4 + 56v^3w + 196v^2w^3 + 252vw^5 + 114w^7)\\
&\quad + 14u_1u(-2u^2v + 2u^2w^2 - 6v^2w - 8vw^3 - 3w^5)\\
&\quad + 7v_1(-u^4 - 12u^2vw - 8u^2w^3 + 4v^3 + 12v^2w^2 + 14vw^4 + 6w^6),\\
S^u(15) &= 7w_1u(u^4 + 8u^2vw + 12u^2w^3 - 4v^3 - 24v^2w^2 - 30vw^4 - 12w^6)\\
&\quad + u_1(35u^4w - 42u^2v^2 + 84u^2vw^2 + 63u^2w^4 - 28v^3w - 56v^2w^3 - 42vw^5 - 12w^7)\\
&\quad + 14v_1u(-2u^2v + 2u^2w^2 - 6v^2w - 8vw^3 - 3w^5),\\
S^v(15) &= 7w_1(4u^4v - u^4w^2 + 12u^2v^2w + 52u^2vw^3 + 30u^2w^5 - 12v^3w^2 - 36v^2w^4 - 42vw^6 - 18w^8)\\
&\quad + 7u_1u(u^4 + 24u^2vw - 4u^2w^3 - 4v^3 + 24v^2w^2 + 34vw^4 + 12w^6)\\
&\quad + v_1(42u^4w - 42u^2v^2 + 168u^2vw^2 + 119u^2w^4 - 56v^3w - 140v^2w^3 - 140vw^5 - 54w^7),
\end{aligned}
$$

2.2.2. *nonlocal Symmetries.*

$$
\begin{aligned}
S^w(4,1) &= -20p_3w_1 - 12p_2u_1 - 6p_1(2w_1w + v_1) + 9u^2 + 16vw + 16w^3\\
S^u(4,1) &= -20p_3u_1 - 12p_2(w_1w + v_1) + 6p_1(w_1u + u_1w) + 8u(3v + w^2),\\
S^v(4,1) &= -20p_3v_1 + 12p_2(w_1u + 3u_1w) + 6p_1(3w_1w^2 + u_1u + 2v_1w) - 39u^2w + 16v^2 - 12w^4,\\
S^w(8,1) &= -1764p_7w_1 - 980p_6u_1 + 1176p_5(-2w_1w - v_1) + 280p_3(w_1v + 3w_1w^2 + u_1u + v_1w)\\
&\quad + 42p_2(2w_1u + 2u_1v + u_1w^2 + 2v_1u) + 14(54u^2v + 9u^2w^2 + 48v^2w + 96vw^3 + 52w^5),\\
S^u(8,1) &= -1764p_7u_1 - 980p_6(w_1w + v_1) + 1176p_5(w_1u + u_1w) + 280p_3(u_1v + v_1u)\\
&\quad + 42p_2(-2w_1u^2 + 2w_1vw + w_1w^3 - 4u_1uw + 2v_1v + v_1w^2)\\
&\quad + 28u(-27u^2w + 36v^2 + 24vw^2 - w^4),\\
S^v(8,1) &= -1764p_7v_1 + 980p_6(w_1u + 3u_1w) + 1176p_5(3w_1w^2 + u_1u + 2v_1w)\\
&\quad + 280p_3(-w_1u^2 - 3w_1w^3 - 4u_1uw + v_1v - v_1w^2)\\
&\quad + 42p_2(-2w_1uv - 7w_1uw^2 - 2u_1u^2 - 6u_1vw - 3u_1w^3 - 6v_1uw)\\
&\quad + 7(-27u^4 - 468u^2vw - 190u^2w^3 + 64v^3 - 144vw^4 - 96w^6),\\
S^w(8,2) &= -2184p_7w_1 - 1456p_6u_1 + 2184p_5(-2w_1w - v_1) + 1040p_3(w_1v + 3w_1w^2 + u_1u + v_1w)\\
&\quad + 312p_2(2w_1uw + 2u_1v + u_1w^2 + 2v_1u)\\
&\quad + 52p_1(-3w_1u^2 + 12w_1vw + 14w_1w^3 - 6u_1uw + 6v_1v + 6v_1w^2),\\
S^u(8,2) &= -2184p_7u_1 - 1456p_6(w_1w + v_1) + 2184p_5(w_1u + u_1w) + 1040p_3(u_1v + v_1u)\\
&\quad + 312p_2(-2w_1u^2 + 2w_1vw + w_1w^3 - 4u_1uw + 2v_1v + v_1w^2)\\
&\quad + 52p_1(-6w_1uv - 12w_1uw^2 - 3u_1u^2 - 6u_1vw - 4u_1w^3 - 6v_1uw),\\
S^v(8,2) &= -2184p_7v_1 + 1456p_6(w_1u + 3u_1w) + 2184p_5(3w_1w^2 + u_1u + 2v_1w)\\
&\quad + 1040p_3(-w_1u^2 - 3w_1w^3 - 4u_1uw + v_1v - v_1w^2)\\
&\quad + 312p_2(-2w_1uv - 7w_1uw^2 - 2u_1u^2 - 6u_1vw - 3u_1w^3 - 6v_1uw)\\
&\quad + 52p_1(6w_1u^2w - 18w_1vw^2 - 18w_1w^4 - 6u_1uv + 12u_1uw^2 - 3v_1u^2 - 12v_1vw - 10v_1w^3),\\
S^w(12,1) &= (1755p_{11}w_1 + 594p_{10}u_1 + 1694p_9(2w_1w + v_1) + 630p_7(-w_1v - 3w_1w^2 - u_1u - v_1w)\\
&\quad + 140p_6(-2w_1uw - 2u_1v - u_1w^2 - 2v_1u)
\end{aligned}
$$

$$
\begin{aligned}
&+ 42p_5(3w_1u^2 - 12w_1vw - 14w_1w^3 + 6u_1uw - 6v_1v - 6v_1w^2)\\
&+ 5p_3(2w_1v^2 + 12w_1vw^2 + 11w_1w^4 + 4u_1uv + 2v_1u^2 + 4v_1vw + 4v_1w^3)\\
&+ 189u^4w - 432u^2v^2 - 144u^2vw^2 + 124u^2w^4 - 256v^3w - 768v^2w^3 - 832vw^5 - 320w^7)/55,\\
S^u(12,1) &= (3510p_{11}u_1 + 1188p_{10}(w_1w + v_1) - 3388p_9(w_1u + u_1w) - 1260p_7(u_1v + v_1u)\\
&+ 280p_6(2w_1u^2 - 2w_1vw - w_1w^3 + 4u_1uw - 2v_1v - v_1w^2)\\
&+ 84p_5(6w_1uv + 12w_1uw^2 + 3u_1u^2 + 6u_1vw + 4u_1w^3 + 6v_1uw)\\
&+ 10p_3(-2w_1u^3 - 4w_1uw^3 - 6u_1u^2w + 2u_1v^2 - u_1w^4 + 4v_1uv)\\
&+ u(81u^4 + 1728u^2vw + 768u^2w^3 - 768v^3 - 768v^2w^2 + 64vw^4 + 192w^6))/110,\\
S^v(12,1) &= (1755p_{11}v_1 + 594p_{10}(-w_1u - 3u_1w) + 1694p_9(-3w_1w^2 - u_1u - 2v_1w)\\
&+ 630p_7(w_1u^2 + 3w_1w^3 + 4u_1uw - v_1v + v_1w^2)\\
&+ 140p_6(2w_1uv + 7w_1uw^2 + 2u_1u^2 + 6u_1vw + 3u_1w^3 + 6v_1uw)\\
&+ 42p_5(-6w_1u^2w + 18w_1vw^2 + 18w_1w^4 + 6u_1uv - 12u_1uw^2 + 3v_1u^2 + 12v_1vw + 10v_1w^3)\\
&+ 5p_3(-4w_1u^2v - 6w_1u^2w^2 - 12w_1vw^3 - 12w_1w^5 - 2u_1u^3 - 16u_1uvw - 4u_1uw^3 - 8v_1u^2w\\
&+ 2v_1v^2 - 4v_1vw^2 - 5v_1w^4)\\
&+ 216u^4v - 567u^4w^2 + 1872u^2v^2w + 1520u^2vw^3 + 188u^2w^5 - 128v^4 + 576v^2w^4\\
&+ 768vw^6 + 312w^8)/55,\\
S^w(12,2) &= (-6240p_{11}w_1 - 2079p_{10}u_1 + 5808p_9(-2w_1w - v_1) + 2016p_7(w_1v + 3w_1w^2 + u_1u + v_1w)\\
&+ 420p_6(2w_1uw + 2u_1v + u_1w^2 + 2v_1u)\\
&+ 112p_5(-3w_1u^2 + 12w_1vw + 14w_1w^3 - 6u_1uw + 6v_1v + 6v_1w^2)\\
&+ p_2(-8w_1u^3 + 24w_1uvw + 12w_1uw^3 - 24u_1u^2w + 12u_1v^2 + 12u_1vw^2 + 3u_1w^4\\
&+ 24v_1uv + 12v_1uw^2)\\
&+ 4(-189u^4w + 432u^2v^2 + 144u^2vw^2 - 124u^2w^4 + 256v^3w + 768v^2w^3 + 832vw^5 + 320w^7))/12,\\
S^u(12,2) &= (-6240p_{11}u_1 - 2079p_{10}(w_1w + v_1) + 5808p_9(w_1u + u_1w)\\
&+ 2016p_7(u_1v + v_1u)\\
&+ 420p_6(-2w_1u^2 + 2w_1vw + w_1w^3 - 4u_1uw + 2v_1v + v_1w^2)\\
&+ 112p_5(-6w_1uv - 12w_1uw^2 - 3u_1u^2 - 6u_1vw - 4u_1w^3 - 6v_1uw)\\
&+ p_2(-24w_1u^2v - 36w_1u^2w^2 + 12w_1v^2w + 12w_1vw^3 + 3w_1w^5 - 8u_1u^3 - 48u_1uvw - 24u_1uw^3\\
&- 24v_1u^2w + 12v_1v^2 + 12v_1vw^2 + 3v_1w^4)\\
&+ 2u(-81u^4 - 1728u^2vw - 768u^2w^3 + 768v^3 + 768v^2w^2 - 64vw^4 - 192w^6))/12,\\
S^v(12,2) &= (-6240p_{11}v_1 + 2079p_{10}(w_1u + 3u_1w) + 5808p_9(3w_1w^2 + u_1u + 2v_1w)\\
&+ 2016p_7(-w_1u^2 - 3w_1w^3 - 4u_1uw + v_1v - v_1w^2)\\
&+ 420p_6(-2w_1uv - 7w_1uw^2 - 2u_1u^2 - 6u_1vw - 3u_1w^3 - 6v_1uw)\\
&+ 112p_5(6w_1u^2w - 18w_1vw^2 - 18w_1w^4 - 6u_1uv + 12u_1uw^2 - 3v_1u^2 - 12v_1vw - 10v_1w^3)\\
&+ p_2(24w_1u^3w - 12w_1uv^2 - 84w_1uvw^2 - 39w_1uw^4 - 24u_1u^2v + 60u_1u^2w^2 - 36u_1v^2w\\
&- 36u_1vw^3 - 9u_1w^5 - 8v_1u^3 - 72v_1uvw - 36v_1uw^3)\\
&+ 4(-216u^4v + 567u^4w^2 - 1872u^2v^2w - 1520u^2vw^3 - 188u^2w^5 + 128v^4\\
&- 576v^2w^4 - 768vw^6 - 312w^8))/12,\\
S^w(12,3) &= (2808p_{11}w_1 + 924p_{10}u_1 + 2541p_9(2w_1w + v_1) + 840p_7(-w_1v - 3w_1w^2 - u_1u - v_1w)
\end{aligned}
$$

$$
\begin{aligned}
&+ 168p_6(-2w_1uw - 2u_1v - u_1w^2 - 2v_1u)\\
&+ 42p_5(3w_1u^2 - 12w_1vw - 14w_1w^3 + 6u_1uw - 6v_1v - 6v_1w^2)\\
&+ p_1(-6w_1u^2v - 12w_1u^2w^2 + 12w_1v^2w + 28w_1vw^3 + 18w_1w^5 - 2u_1u^3\\
&- 12u_1uvw - 8u_1uw^3 - 6v_1u^2w + 6v_1v^2 + 12v_1vw^2 + 7v_1w^4)\\
&+ 2(189u^4w - 432u^2v^2 - 144u^2vw^2 + 124u^2w^4 - 256v^3w - 768v^2w^3 - 832vw^5 - 320w^7))/18,\\
S^u(12,3) = {}&(2808p_{11}u_1 + 924p_{10}(w_1w + v_1) - 2541p_9(w_1u + u_1w) - 840p_7(u_1v + v_1u)\\
&+ 168p_6(2w_1u^2 - 2w_1vw - w_1w^3 + 4u_1uw - 2v_1v - v_1w^2)\\
&+ 42p_5(6w_1uv + 12w_1uw^2 + 3u_1u^2 + 6u_1vw + 4u_1w^3 + 6v_1uw)\\
&+ p_1(4w_1u^3w - 6w_1uv^2 - 24w_1uvw^2 - 15w_1uw^4 - 6u_1u^2v + 6u_1u^2w^2 - 6u_1v^2w\\
&- 8u_1vw^3 - 3u_1w^5 - 2v_1u^3 - 12v_1uvw - 8v_1uw^3)\\
&+ u(81u^4 + 1728u^2vw + 768u^2w^3 - 768v^3 - 768v^2w^2 + 64vw^4 + 192w^6))/18,\\
S^v(12,3) = {}&(2808p_{11}v_1 + 924p_{10}(-w_1u - 3u_1w) + 2541p_9(-3w_1w^2 - u_1u - 2v_1w)\\
&+ 840p_7(w_1u^2 + 3w_1w^3 + 4u_1uw - v_1v + v_1w^2)\\
&+ 168p_6(2w_1uv + 7w_1uw^2 + 2u_1u^2 + 6u_1vw + 3u_1w^3 + 6v_1uw)\\
&+ 42p_5(-6w_1u^2w + 18w_1vw^2 + 18w_1w^4 + 6u_1uv - 12u_1uw^2 + 3v_1u^2 + 12v_1vw + 10v_1w^3)\\
&+ p_1(2w_1u^4 + 12w_1u^2vw + 26w_1u^2w^3 - 18w_1v^2w^2 - 36w_1vw^4 - 21w_1w^6 + 12u_1u^3w\\
&- 6u_1uv^2 + 24u_1uvw^2 + 17u_1uw^4 - 6v_1u^2v + 12v_1u^2w^2 - 12v_1v^2w - 20v_1vw^3 - 10v_1w^5)\\
&+ 2(216u^4v - 567u^4w^2 + 1872u^2v^2w + 1520u^2vw^3 + 188u^2w^5 - 128v^4\\
&+ 576v^2w^4 + 768vw^6 + 312w^8))/18,\\
S^w(16,1) = {}&(50388p_{15}w_1\\
&+ 8580p_{14}u_1\\
&+ 60060p_{13}(2w_1w + v_1)\\
&+ 28080p_{11}(-w_1v - 3w_1w^2 - u_1u - v_1w)\\
&+ 4158p_{10}(-2w_1uw - 2u_1v - u_1w^2 - 2v_1u)\\
&+ 3388p_9(3w_1u^2 - 12w_1vw - 14w_1w^3 + 6u_1uw - 6v_1v - 6v_1w^2)\\
&+ 1260p_7(2w_1v^2 + 12w_1vw^2 + 11w_1w^4 + 4u_1uv + 2v_1u^2 + 4v_1vw + 4v_1w^3)\\
&+ 70p_6(-8w_1u^3 + 24w_1uvw + 12w_1uw^3 - 24u_1u^2w + 12u_1v^2 + 12u_1vw^2 + 3u_1w^4\\
&+ 24v_1uv + 12v_1uw^2)\\
&+ 84p_5(-6w_1u^2v - 12w_1u^2w^2 + 12w_1v^2w + 28w_1vw^3 + 18w_1w^5 - 2u_1u^3\\
&- 12u_1uvw - 8u_1uw^3 - 6v_1u^2w + 6v_1v^2 + 12v_1vw^2 + 7v_1w^4)\\
&+ 243u^6 + 7560u^4vw + 3060u^4w^3 - 5760u^2v^3 - 2880u^2v^2w^2 + 4960u^2vw^4 + 3312u^2w^6\\
&- 2560v^4w - 10240v^3w^3 - 16640v^2w^5 - 12800vw^7 - 3872w^9)/1512,\\
S^u(16,1) = {}&(25194p_{15}u_1\\
&+ 4290p_{14}(w_1w + v_1)\\
&- 30030p_{13}(w_1u + u_1w)\\
&- 14040p_{11}(u_1v + v_1u)\\
&+ 2079p_{10}(2w_1u^2 - 2w_1vw - w_1w^3 + 4u_1uw - 2v_1v - v_1w^2)\\
&+ 1694p_9(6w_1uv + 12w_1uw^2 + 3u_1u^2 + 6u_1vw + 4u_1w^3 + 6v_1uw)
\end{aligned}
$$

$$
\begin{aligned}
&+ 630p_7(-2w_1u^3 - 4w_1uw^3 - 6u_1u^2w + 2u_1v^2 - u_1w^4 + 4v_1uv)\\
&+ 35p_6(-24w_1u^2v - 36w_1u^2w^2 + 12w_1v^2w + 12w_1vw^3 + 3w_1w^5 - 8u_1u^3\\
&- 48u_1uvw - 24u_1uw^3 - 24v_1u^2w + 12v_1v^2 + 12v_1vw^2 + 3v_1w^4)\\
&+ 42p_5(4w_1u^3w - 6w_1uv^2 - 24w_1uvw^2 - 15w_1uw^4 - 6u_1u^2v + 6u_1u^2w^2 - 6u_1v^2w\\
&- 8u_1vw^3 - 3u_1w^5 - 2v_1u^3 - 12v_1uvw - 8v_1uw^3)\\
&+ u(810u^4v - 1431u^4w^2 + 8640u^2v^2w + 7680u^2vw^3 + 1456u^2w^5 - 1920v^4\\
&- 2560v^3w^2 + 320v^2w^4 + 1920vw^6 + 808w^8))/756,\\
S^v(16,1) = {}&(50388p_{15}v_1\\
&+ 8580p_{14}(-w_1u - 3u_1w)\\
&+ 60060p_{13}(-3w_1w^2 - u_1u - 2v_1w)\\
&+ 28080p_{11}(w_1u^2 + 3w_1w^3 + 4u_1uw - v_1v + v_1w^2)\\
&+ 4158p_{10}(2w_1uv + 7w_1uw^2 + 2u_1u^2 + 6u_1vw + 3u_1w^3 + 6v_1uw)\\
&+ 3388p_9(-6w_1u^2w + 18w_1vw^2 + 18w_1w^4 + 6u_1uv - 12u_1uw^2 + 3v_1u^2 + 12v_1vw + 10v_1w^3)\\
&+ 1260p_7(-4w_1u^2v - 6w_1u^2w^2 - 12w_1vw^3 - 12w_1w^5 - 2u_1u^3 - 16u_1uvw\\
&- 4u_1uw^3 - 8v_1u^2w + 2v_1v^2 - 4v_1vw^2 - 5v_1w^4)\\
&+ 70p_6(24w_1u^3w - 12w_1uv^2 - 84w_1uvw^2 - 39w_1uw^4 - 24u_1u^2v + 60u_1u^2w^2\\
&- 36u_1v^2w - 36u_1vw^3 - 9u_1w^5 - 8v_1u^3 - 72v_1uvw - 36v_1uw^3)\\
&+ 84p_5(2w_1u^4 + 12w_1u^2vw + 26w_1u^2w^3 - 18w_1v^2w^2 - 36w_1vw^4 - 21w_1w^6 + 12u_1u^3w\\
&- 6u_1uv^2 + 24u_1uvw^2 + 17u_1uw^4 - 6v_1u^2v + 12v_1u^2w^2 - 12v_1v^2w - 20v_1vw^3 - 10v_1w^5)\\
&- 2025u^6w + 4320u^4v^2 - 22680u^4vw^2 - 11460u^4w^4 + 24960u^2v^3w + 30400u^2v^2w^3 + 7520u^2vw^5\\
&- 1680u^2w^7 - 1024v^5 + 7680v^3w^4 + 15360v^2w^6 + 12480vw^8 + 3840w^10)/1512,\\
S^w(16,2) = {}&(125970p_{15}w_1\\
&+ 21060p_{14}u_1\\
&+ 144144p_{13}(2w_1w + v_1)\\
&+ 63180p_{11}(-w_1v - 3w_1w^2 - u_1u - v_1w)\\
&+ 8910p_{10}(-2w_1uw - 2u_1v - u_1w^2 - 2v_1u)\\
&+ 6776p_9(3w_1u^2 - 12w_1vw - 14w_1w^3 + 6u_1uw - 6v_1v - 6v_1w^2)\\
&+ 1890p_7(2w_1v^2 + 12w_1vw^2 + 11w_1w^4 + 4u_1uv + 2v_1u^2 + 4v_1vw + 4v_1w^3)\\
&+ 70p_6(-8w_1u^3 + 24w_1uvw + 12w_1uw^3 - 24u_1u^2w + 12u_1v^2 + 12u_1vw^2 + 3u_1w^4\\
&+ 24v_1uv + 12v_1uw^2)\\
&+ 2p_3(-15w_1u^4 - 60w_1u^2w^3 + 20w_1v^3 + 180w_1v^2w^2 + 330w_1vw^4 + 182w_1w^6 - 60u_1u^3w\\
&+ 60u_1uv^2 - 30u_1uw^4 + 60v_1u^2v + 60v_1v^2w + 120v_1vw^3 + 66v_1w^5)\\
&+ 3(243u^6 + 7560u^4vw + 3060u^4w^3 - 5760u^2v^3 - 2880u^2v^2w^2 + 4960u^2vw^4 + 3312u^2w^6\\
&- 2560v^4w - 10240v^3w^3 - 16640v^2w^5 - 12800vw^7 - 3872w^9))/364,\\
S^u(16,2) = {}&(62985p_{15}u_1\\
&+ 10530p_{14}(w_1w + v_1)\\
&- 72072p_{13}(w_1u + u_1w)\\
&- 31590p_{11}(u_1v + v_1u)
\end{aligned}
$$

$$
\begin{aligned}
&+ 4455p_{10}(2w_1u^2 - 2w_1vw - w_1w^3 + 4u_1uw - 2v_1v - v_1w^2)\\
&+ 3388p_9(6w_1uv + 12w_1uw^2 + 3u_1u^2 + 6u_1vw + 4u_1w^3 + 6v_1uw)\\
&+ 945p_7(-2w_1u^3 - 4w_1uw^3 - 6u_1u^2w + 2u_1v^2 - u_1w^4 + 4v_1uv)\\
&+ 35p_6(-24w_1u^2v - 36w_1u^2w^2 + 12w_1v^2w + 12w_1vw^3\\
&+ 3w_1w^5 - 8u_1u^3\\
&- 48u_1uvw - 24u_1uw^3 - 24v_1u^2w + 12v_1v^2 + 12v_1vw^2 + 3v_1w^4)\\
&+ p_3(-60w_1u^3v - 60w_1u^3w^2 - 120w_1uvw^3 - 96w_1uw^5 - 15u_1u^4 - 180u_1u^2vw\\
&- 60u_1u^2w^3 + 20u_1v^3 - 30u_1vw^4 - 16u_1w^6 - 60v_1u^3w + 60v_1uv^2 - 30v_1uw^4)\\
&+ 3u(810u^4v - 1431u^4w^2 + 8640u^2v^2w + 7680u^2vw^3 + 1456u^2w^5 - 1920v^4 - 2560v^3w^2\\
&+ 320v^2w^4 + 1920vw^6 + 808w^8))/182,\\
S^v(16,2) &= (125970p_{15}v_1\\
&+ 21060p_{14}(-w_1u - 3u_1w)\\
&+ 144144p_{13}(-3w_1w^2 - u_1u - 2v_1w)\\
&+ 63180p_{11}(w_1u^2 + 3w_1w^3 + 4u_1uw - v_1v + v_1w^2)\\
&+ 8910p_{10}(2w_1uv + 7w_1uw^2 + 2u_1u^2 + 6u_1vw + 3u_1w^3 + 6v_1uw)\\
&+ 6776p_9(-6w_1u^2w + 18w_1vw^2 + 18w_1w^4 + 6u_1uv - 12u_1uw^2 + 3v_1u^2 + 12v_1vw + 10v_1w^3)\\
&+ 1890p_7(-4w_1u^2v - 6w_1u^2w^2 - 12w_1vw^3 - 12w_1w^5 - 2u_1u^3 - 16u_1uvw\\
&- 4u_1uw^3 - 8v_1u^2w + 2v_1v^2 - 4v_1vw^2 - 5v_1w^4)\\
&+ 70p_6(24w_1u^3w - 12w_1uv^2 - 84w_1uvw^2 - 39w_1uw^4 - 24u_1u^2v + 60u_1u^2w^2\\
&- 36u_1v^2w - 36u_1vw^3 - 9u_1w^5 - 8v_1u^3 - 72v_1uvw - 36v_1uw^3)\\
&+ 2p_3(60w_1u^4w - 60w_1u^2v^2 - 180w_1u^2vw^2 + 30w_1u^2w^4 - 180w_1v^2w^3 - 360w_1vw^5\\
&- 198w_1w^7 - 60u_1u^3v + 180u_1u^3w^2 - 240u_1uv^2w - 120u_1uvw^3 + 24u_1uw^5\\
&- 15v_1u^4 - 240v_1u^2vw - 60v_1u^2w^3 + 20v_1v^3 - 60v_1v^2w^2 - 150v_1vw^4 - 82v_1w^6)\\
&+ 3(-2025u^6w + 4320u^4v^2 - 22680u^4vw^2 - 11460u^4w^4 + 24960u^2v^3w\\
&+ 30400u^2v^2w^3 + 7520u^2vw^5 - 1680u^2w^7 - 1024v^5 + 7680v^3w^4 + 15360v^2w^6\\
&+ 12480vw^8 + 3840w^10))/364,\\
S^w(16,3) &= (-155040p_{15}w_1\\
&- 25740p_{14}u_1\\
&+ 174720p_{13}(-2w_1w - v_1)\\
&+ 74880p_{11}(w_1v + 3w_1w^2 + u_1u + v_1w)\\
&+ 10395p_{10}(2w_1uw + 2u_1v + u_1w^2 + 2v_1u)\\
&+ 7744p_9(-3w_1u^2 + 12w_1vw + 14w_1w^3 - 6u_1uw + 6v_1v + 6v_1w^2)\\
&+ 2016p_7(-2w_1v^2 - 12w_1vw^2 - 11w_1w^4 - 4u_1uv - 2v_1u^2 - 4v_1vw - 4v_1w^3)\\
&+ 70p_6(8w_1u^3 - 24w_1uvw - 12w_1uw^3 + 24u_1u^2w - 12u_1v^2 - 12u_1vw^2 - 3u_1w^4\\
&- 24v_1uv - 12v_1uw^2)\\
&+ 3p_2(-16w_1u^3v - 24w_1u^3w^2 + 24w_1uv^2w + 24w_1uvw^3 + 6w_1uw^5 - 4u_1u^4\\
&- 48u_1u^2vw - 24u_1u^2w^3 + 8u_1v^3 + 12u_1v^2w^2 + 6u_1vw^4 + u_1w^6 - 16v_1u^3w\\
&+ 24v_1uv^2 + 24v_1uvw^2 + 6v_1uw^4)
\end{aligned}
$$

$$
\begin{aligned}
&+4(-243u^6-7560u^4vw-3060u^4w^3+5760u^2v^3+2880u^2v^2w^2-4960u^2vw^4-3312u^2w^6\\
&+2560v^4w+10240v^3w^3+16640v^2w^5+12800vw^7+3872w^9))/18,\\
S^u(16,3)&=(-155040p_{15}u_1\\
&-25740p_{14}(w_1w+v_1)\\
&+174720p_{13}(w_1u+u_1w)\\
&+74880p_{11}(u_1v+v_1u)\\
&+10395p_{10}(-2w_1u^2+2w_1vw+w_1w^3-4u_1uw+2v_1v+v_1w^2)\\
&+7744p_9(-6w_1uv-12w_1uw^2-3u_1u^2-6u_1vw-4u_1w^3-6v_1uw)\\
&+2016p_7(2w_1u^3+4w_1uw^3+6u_1u^2w-2u_1v^2+u_1w^4-4v_1uv)\\
&+70p_6(24w_1u^2v+36w_1u^2w^2-12w_1v^2w-12w_1vw^3-3w_1w^5+8u_1u^3+48u_1uvw\\
&+24u_1uw^3+24v_1u^2w-12v_1v^2-12v_1vw^2-3v_1w^4)\\
&+3p_2(12w_1u^4w-24w_1u^2v^2-72w_1u^2vw^2-30w_1u^2w^4+8w_1v^3w+12w_1v^2w^3\\
&+6w_1vw^5+w_1w^7-16u_1u^3v+24u_1u^3w^2-48u_1uv^2w-48u_1uvw^3-12u_1uw^5\\
&-4v_1u^4-48v_1u^2vw-24v_1u^2w^3+8v_1v^3+12v_1v^2w^2+6v_1vw^4+v_1w^6)\\
&+8u(-810u^4v+1431u^4w^2-8640u^2v^2w-7680u^2vw^3-1456u^2w^5+1920v^4+2560v^3w^2\\
&-320v^2w^4-1920vw^6-808w^8))/18,\\
S^v(16,3)&=(-155040p_{15}v_1\\
&+25740p_{14}(w_1u+3u_1w)\\
&+174720p_{13}(3w_1w^2+u_1u+2v_1w)\\
&+74880p_{11}(-w_1u^2-3w_1w^3-4u_1uw+v_1v-v_1w^2)\\
&+10395p_{10}(-2w_1uv-7w_1uw^2-2u_1u^2-6u_1vw-3u_1w^3-6v_1uw)\\
&+7744p_9(6w_1u^2w-18w_1vw^2-18w_1w^4-6u_1uv+12u_1uw^2-3v_1u^2-12v_1vw-10v_1w^3)\\
&+2016p_7(4w_1u^2v+6w_1u^2w^2+12w_1vw^3+12w_1w^5+2u_1u^3+16u_1uvw+4u_1uw^3\\
&+8v_1u^2w-2v_1v^2+4v_1vw^2+5v_1w^4)\\
&+70p_6(-24w_1u^3w+12w_1uv^2+84w_1uvw^2+39w_1uw^4+24u_1u^2v-60u_1u^2w^2\\
&+36u_1v^2w+36u_1vw^3+9u_1w^5+8v_1u^3+72v_1uvw+36v_1uw^3)\\
&+3p_2(4w_1u^5+48w_1u^3vw+72w_1u^3w^3-8w_1uv^3-84w_1uv^2w^2-78w_1uvw^4\\
&-19w_1uw^6+28u_1u^4w-24u_1u^2v^2+120u_1u^2vw^2+66u_1u^2w^4-24u_1v^3w-36u_1v^2w^3\\
&-18u_1vw^5-3u_1w^7-16v_1u^3v+40v_1u^3w^2-72v_1uv^2w-72v_1uvw^3-18v_1uw^5)\\
&+4(2025u^6w-4320u^4v^2+22680u^4vw^2+11460u^4w^4-24960u^2v^3w-30400u^2v^2w^3-7520u^2vw^5\\
&+1680u^2w^7+1024v^5-7680v^3w^4-15360v^2w^6-12480vw^8-3840w^10))/18,\\
S^w(16,4)&=(251940p_{15}w_1\\
&+41580p_{14}u_1\\
&+280280p_{13}(2w_1w+v_1)\\
&+117936p_{11}(-w_1v-3w_1w^2-u_1u-v_1w)\\
&+16170p_{10}(-2w_1uw-2u_1v-u_1w^2-2v_1u)\\
&+11858p_9(3w_1u^2-12w_1vw-14w_1w^3+6u_1uw-6v_1v-6v_1w^2)\\
&+2940p_7(2w_1v^2+12w_1vw^2+11w_1w^4+4u_1uv+2v_1u^2+4v_1vw+4v_1w^3)
\end{aligned}
$$

$$
\begin{aligned}
&+ 98p_6(-8w_1u^3 + 24w_1uvw + 12w_1uw^3 - 24u_1u^2w + 12u_1v^2 + 12u_1vw^2 + 3u_1w^4\\
&+ 24v_1uv + 12v_1uw^2)\\
&+ 2p_1(14w_1u^4w - 42w_1u^2v^2 - 168w_1u^2vw^2 - 105w_1u^2w^4 + 56w_1v^3w + 196w_1v^2w^3\\
&+ 252w_1vw^5 + 114w_1w^7 - 28u_1u^3v + 28u_1u^3w^2 - 84u_1uv^2w - 112u_1uvw^3 - 42u_1uw^5\\
&- 7v_1u^4 - 84v_1u^2vw - 56v_1u^2w^3 + 28v_1v^3 + 84v_1v^2w^2 + 98v_1vw^4 + 42v_1w^6)\\
&+ 7(243u^6 + 7560u^4vw + 3060u^4w^3 - 5760u^2v^3 - 2880u^2v^2w^2 + 4960u^2vw^4 + 3312u^2w^6\\
&- 2560v^4w - 10240v^3w^3 - 16640v^2w^5 - 12800vw^7 - 3872w^9))/228,
\end{aligned}
$$

$$
\begin{aligned}
S^u(16,4) = {}&(125970p_{15}u_1\\
&+ 20790p_{14}(w_1w + v_1)\\
&- 140140p_{13}(w_1u + u_1w)\\
&- 58968p_{11}(u_1v + v_1u)\\
&+ 8085p_{10}(2w_1u^2 - 2w_1vw - w_1w^3 + 4u_1uw - 2v_1v - v_1w^2)\\
&+ 5929p_9(6w_1uv + 12w_1uw^2 + 3u_1u^2 + 6u_1vw + 4u_1w^3 + 6v_1uw)\\
&+ 1470p_7(-2w_1u^3 - 4w_1uw^3 - 6u_1u^2w + 2u_1v^2 - u_1w^4 + 4v_1uv)\\
&+ 49p_6(-24w_1u^2v - 36w_1u^2w^2 + 12w_1v^2w + 12w_1vw^3 + 3w_1w^5 - 8u_1u^3\\
&- 48u_1uvw - 24u_1uw^3 - 24v_1u^2w + 12v_1v^2 + 12v_1vw^2 + 3v_1w^4)\\
&+ p_1(7w_1u^5 + 56w_1u^3vw + 84w_1u^3w^3 - 28w_1uv^3 - 168w_1uv^2w^2 - 210w_1uvw^4\\
&- 84w_1uw^6 + 35u_1u^4w - 42u_1u^2v^2 + 84u_1u^2vw^2 + 63u_1u^2w^4 - 28u_1v^3w - 56u_1v^2w^3\\
&- 42u_1vw^5 - 12u_1w^7 - 28v_1u^3v + 28v_1u^3w^2 - 84v_1uv^2w - 112v_1uvw^3 - 42v_1uw^5)\\
&+ 7u(810u^4v - 1431u^4w^2 + 8640u^2v^2w + 7680u^2vw^3 + 1456u^2w^5 - 1920v^4 - 2560v^3w^2\\
&+ 320v^2w^4 + 1920vw^6 + 808w^8))/114,
\end{aligned}
$$

$$
\begin{aligned}
S^v(16,4) = {}&(251940p_{15}v_1\\
&+ 41580p_{14}(-w_1u - 3u_1w)\\
&+ 280280p_{13}(-3w_1w^2 - u_1u - 2v_1w)\\
&+ 117936p_{11}(w_1u^2 + 3w_1w^3 + 4u_1uw - v_1v + v_1w^2)\\
&+ 16170p_{10}(2w_1uv + 7w_1uw^2 + 2u_1u^2 + 6u_1vw + 3u_1w^3 + 6v_1uw)\\
&+ 11858p_9(-6w_1u^2w + 18w_1vw^2 + 18w_1w^4 + 6u_1uv - 12u_1uw^2 + 3v_1u^2 + 12v_1vw + 10v_1w^3)\\
&+ 2940p_7(-4w_1u^2v - 6w_1u^2w^2 - 12w_1vw^3 - 12w_1w^5 - 2u_1u^3 - 16u_1uvw\\
&- 4u_1uw^3 - 8v_1u^2w + 2v_1v^2 - 4v_1vw^2 - 5v_1w^4)\\
&+ 98p_6(24w_1u^3w - 12w_1uv^2 - 84w_1uvw^2 - 39w_1uw^4 - 24u_1u^2v + 60u_1u^2w^2\\
&- 36u_1v^2w - 36u_1vw^3 - 9u_1w^5 - 8v_1u^3 - 72v_1uvw - 36v_1uw^3)\\
&+ 2p_1(28w_1u^4v - 7w_1u^4w^2 + 84w_1u^2v^2w + 364w_1u^2vw^3 + 210w_1u^2w^5 - 84w_1v^3w^2\\
&- 252w_1v^2w^4 - 294w_1vw^6 - 126w_1w^8 + 7u_1u^5 + 168u_1u^3vw - 28u_1u^3w^3 - 28u_1uv^3\\
&+ 168u_1uv^2w^2 + 238u_1uvw^4 + 84u_1uw^6 + 42v_1u^4w - 42v_1u^2v^2 + 168v_1u^2vw^2 + 119v_1u^2w^4\\
&- 56v_1v^3w - 140v_1v^2w^3 - 140v_1vw^5 - 54v_1w^7)\\
&+ 7(-2025u^6w + 4320u^4v^2 - 22680u^4vw^2 - 11460u^4w^4 + 24960u^2v^3w + 30400u^2v^2w^3\\
&+ 7520u^2vw^5 - 1680u^2w^7 - 1024v^5 + 7680v^3w^4 + 15360v^2w^6 + 12480vw^8 + 3840w^10))/228
\end{aligned}
$$

The presented results above indicate that at each degree $4, 8, 12, 16, \ldots$ a new hierarchy starts.

2.2.3. $(x,t)$ - *dependent symmetries.*

$$\begin{aligned}
S^w(0,1) &= xw_1 - 2w,\\
S^u(0,1) &= xu_1 - 3u,\\
S^v(0,1) &= xv_1 - 4v,\\
S^w(0,2) &= tu_1 + 2w,\\
S^u(0,2) &= t(w_1 w + v_1) + 3u,\\
S^v(0,2) &= t(-w_1 u - 3u_1 w) + 4v,\\
S^w(4,1) &= x(w_1 v + 3w_1 w^2 + u_1 u + v_1 w) + 3p_1(-2w_1 w - v_1) - 4p_2 u_1 - 5p_3 w_1,\\
S^u(4,1) &= x(u_1 v + v_1 u) + 3p_1(w_1 u + u_1 w) - 4p_2(w_1 w + v_1) - 5p_3 u_1,\\
S^v(4,1) &= x(-w_1 u^2 - 3w_1 w^3 - 4u_1 uw + v_1 v - v_1 w^2) + 3p_1(3w_1 w^2 + u_1 u + 2v_1 w)\\
&\quad + 4p_2(w_1 u + 3u_1 w) - 5p_3 v_1,\\
S^w(4,2) &= t(2w_1 uw + 2u_1 v + u_1 w^2 + 2v_1 u) + 4p_1(2w_1 w + v_1) + 6p_2 u_1 + 8p_3 w_1,\\
S^u(4,2) &= t(-2w_1 u^2 + 2w_1 vw + w_1 w^3 - 4u_1 uw + 2v_1 v + v_1 w^2) - 4p_1(w_1 u + u_1 w)\\
&\quad + 6p_2(w_1 w + v_1) + 8p_3 u_1,\\
S^v(4,2) &= t(-2w_1 uv - 7w_1 uw^2 - 2u_1 u^2 - 6u_1 vw - 3u_1 w^3 - 6v_1 uw)\\
&\quad + 4p_1(-3w_1 w^2 - u_1 u - 2v_1 w) + 6p_2(-w_1 u - 3u_1 w) + 8p_3 v_1,\\
S^w(8,1) &= x(2w_1 v^2 + 12w_1 vw^2 + 11w_1 w^4 + 4u_1 uv + 2v_1 u^2 + 4v_1 vw + 4v_1 w^3)\\
&\quad + 4p_2(2w_1 uw + 2u_1 v + u_1 w^2 + 2v_1 u)\\
&\quad + 20p_3(w_1 v + 3w_1 w^2 + u_1 u + v_1 w)\\
&\quad + 56p_5(-2w_1 w - v_1)\\
&\quad - 40p_6 u_1 - 63p_7 w_1,\\
S^u(8,1) &= x(-2w_1 u^3 - 4w_1 uw^3 - 6u_1 u^2 w + 2u_1 v^2 - u_1 w^4 + 4v_1 uv)\\
&\quad + 4p_2(-2w_1 u^2 + 2w_1 vw + w_1 w^3 - 4u_1 uw + 2v_1 v + v_1 w^2)\\
&\quad + 20p_3(u_1 v + v_1 u)\\
&\quad + 56p_5(w_1 u + u_1 w)\\
&\quad - 40p_6(w_1 w + v_1)\\
&\quad - 63p_7 u_1,\\
S^v(8,1) &= x(-4w_1 u^2 v - 6w_1 u^2 w^2 - 12w_1 vw^3 - 12w_1 w^5 - 2u_1 u^3 - 16u_1 uvw\\
&\quad - 4u_1 uw^3 - 8v_1 u^2 w + 2v_1 v^2 - 4v_1 vw^2 - 5v_1 w^4)\\
&\quad + 4p_2(-2w_1 uv - 7w_1 uw^2 - 2u_1 u^2 - 6u_1 vw - 3u_1 w^3 - 6v_1 uw)\\
&\quad + 20p_3(-w_1 u^2 - 3w_1 w^3 - 4u_1 uw + v_1 v - v_1 w^2)\\
&\quad + 56p_5(3w_1 w^2 + u_1 u + 2v_1 w)\\
&\quad + 40p_6(w_1 u + 3u_1 w)\\
&\quad - 63p_7 v_1,\\
S^w(8,2) &= t(-8w_1 u^3 + 24w_1 uvw + 12w_1 uw^3 - 24u_1 u^2 w + 12u_1 v^2 + 12u_1 vw^2 + 3u_1 w^4\\
&\quad + 24v_1 uv + 12v_1 uw^2)\\
&\quad + 12p_2(-2w_1 uw - 2u_1 v - u_1 w^2 - 2v_1 u)
\end{aligned}$$

$$
\begin{aligned}
&+ 64p_3(-w_1v - 3w_1w^2 - u_1u - v_1w)\\
&+ 192p_5(2w_1w + v_1)\\
&+ 140p_6u_1\\
&+ 224p_7w_1,\\
S^u(8,2) &= t(-24w_1u^2v - 36w_1u^2w^2 + 12w_1v^2w + 12w_1vw^3 + 3w_1w^5 - 8u_1u^3 - 48u_1uvw\\
&- 24u_1uw^3 - 24v_1u^2w + 12v_1v^2 + 12v_1vw^2 + 3v_1w^4)\\
&+ 12p_2(2w_1u^2 - 2w_1vw - w_1w^3 + 4u_1uw - 2v_1v - v_1w^2)\\
&- 64p_3(u_1v + v_1u)\\
&- 192p_5(w_1u + u_1w)\\
&+ 140p_6(w_1w + v_1)\\
&+ 224p_7u_1,\\
S^v(8,2) &= t(24w_1u^3w - 12w_1uv^2 - 84w_1uvw^2 - 39w_1uw^4 - 24u_1u^2v + 60u_1u^2w^2\\
&- 36u_1v^2w - 36u_1vw^3 - 9u_1w^5 - 8v_1u^3 - 72v_1uvw - 36v_1uw^3)\\
&+ 12p_2(2w_1uv + 7w_1uw^2 + 2u_1u^2 + 6u_1vw + 3u_1w^3 + 6v_1uw)\\
&+ 64p_3(w_1u^2 + 3w_1w^3 + 4u_1uw - v_1v + v_1w^2)\\
&+ 192p_5(-3w_1w^2 - u_1u - 2v_1w)\\
&+ 140p_6(-w_1u - 3u_1w)\\
&+ 224p_7v_1,\\
S^w(12,1) &= x(-15w_1u^4 - 60w_1u^2w^3 + 20w_1v^3 + 180w_1v^2w^2 + 330w_1vw^4 + 182w_1w^6\\
&- 60u_1u^3w + 60u_1uv^2 - 30u_1uw^4 + 60v_1u^2v + 60v_1v^2w + 120v_1vw^3 + 66v_1w^5)\\
&+ 140p_5(3w_1u^2 - 12w_1vw - 14w_1w^3 + 6u_1uw - 6v_1v - 6v_1w^2)\\
&+ 600p_6(-2w_1uw - 2u_1v - u_1w^2 - 2v_1u)\\
&+ 3150p_7(-w_1v - 3w_1w^2 - u_1u - v_1w)\\
&+ 10164p_9(2w_1w + v_1)\\
&+ 3780p_{10}u_1 + 11700p_{11}w_1 + 10(189u^4w - 432u^2v^2 - 144u^2vw^2 + 124u^2w^4\\
&- 256v^3w - 768v^2w^3 - 832vw^5 - 320w^7),\\
S^u(12,1) &= x(-60w_1u^3v - 60w_1u^3w^2 - 120w_1uvw^3 - 96w_1uw^5 - 15u_1u^4 - 180u_1u^2vw\\
&- 60u_1u^2w^3 + 20u_1v^3 - 30u_1vw^4 - 16u_1w^6 - 60v_1u^3w + 60v_1uv^2 - 30v_1uw^4)\\
&+ 140p_5(6w_1uv + 12w_1uw^2 + 3u_1u^2 + 6u_1vw + 4u_1w^3 + 6v_1uw)\\
&+ 600p_6(2w_1u^2 - 2w_1vw - w_1w^3 + 4u_1uw - 2v_1v - v_1w^2)\\
&- 3150p_7(u_1v + v_1u)\\
&- 10164p_9(w_1u + u_1w)\\
&+ 3780p_{10}(w_1w + v_1)\\
&+ 11700p_{11}u_1 + 5u(81u^4 + 1728u^2vw + 768u^2w^3 - 768v^3 - 768v^2w^2 + 64vw^4 + 192w^6),\\
S^v(12,1) &= x(60w_1u^4w - 60w_1u^2v^2 - 180w_1u^2vw^2 + 30w_1u^2w^4 - 180w_1v^2w^3 - 360w_1vw^5\\
&- 198w_1w^7 - 60u_1u^3v + 180u_1u^3w^2 - 240u_1uv^2w - 120u_1uvw^3 + 24u_1uw^5 - 15v_1u^4\\
&- 240v_1u^2vw - 60v_1u^2w^3 + 20v_1v^3 - 60v_1v^2w^2 - 150v_1vw^4 - 82v_1w^6)\\
&+ 140p_5(-6w_1u^2w + 18w_1vw^2 + 18w_1w^4 + 6u_1uv - 12u_1uw^2 + 3v_1u^2 + 12v_1vw + 10v_1w^3)\\
&+ 600p_6(2w_1uv + 7w_1uw^2 + 2u_1u^2 + 6u_1vw + 3u_1w^3 + 6v_1uw)
\end{aligned}
$$

$$
\begin{aligned}
&+ 3150p_7(w_1u^2 + 3w_1w^3 + 4u_1uw - v_1v + v_1w^2)\\
&+ 10164p_9(-3w_1w^2 - u_1u - 2v_1w)\\
&+ 3780p_{10}(-w_1u - 3u_1w)\\
&+ 11700p_{11}v_1\\
&+ 10(216u^4v - 567u^4w^2 + 1872u^2v^2w + 1520u^2vw^3 + 188u^2w^5 - 128v^4 + 576v^2w^4\\
&+ 768vw^6 + 312w^8),
\end{aligned}
$$

$$
\begin{aligned}
S^w(12,2) = {}&(5t(-16w_1u^3v - 24w_1u^3w^2 + 24w_1uv^2w + 24w_1uvw^3 + 6w_1uw^5 - 4u_1u^4 - 48u_1u^2vw\\
&- 24u_1u^2w^3 + 8u_1v^3 + 12u_1v^2w^2 + 6u_1vw^4 + u_1w^6 - 16v_1u^3w + 24v_1uv^2 + 24v_1uvw^2 + 6v_1uw^4)\\
&+ 96p_5(-3w_1u^2 + 12w_1vw + 14w_1w^3 - 6u_1uw + 6v_1v + 6v_1w^2)\\
&+ 420p_6(2w_1uw + 2u_1v + u_1w^2 + 2v_1u)\\
&+ 2240p_7(w_1v + 3w_1w^2 + u_1u + v_1w)\\
&+ 7392p_9(-2w_1w - v_1)\\
&- 2772p_{10}u_1 - 8640p_{11}w_1\\
&+ 8(-189u^4w + 432u^2v^2 + 144u^2vw^2 - 124u^2w^4 + 256v^3w + 768v^2w^3 + 832vw^5 + 320w^7))/5,
\end{aligned}
$$

$$
\begin{aligned}
S^u(12,2) = {}&(5t(12w_1u^4w - 24w_1u^2v^2 - 72w_1u^2vw^2 - 30w_1u^2w^4 + 8w_1v^3w + 12w_1v^2w^3\\
&+ 6w_1vw^5 + w_1w^7 - 16u_1u^3v + 24u_1u^3w^2 - 48u_1uv^2w - 48u_1uvw^3 - 12u_1uw^5\\
&- 4v_1u^4 - 48v_1u^2vw - 24v_1u^2w^3 + 8v_1v^3 + 12v_1v^2w^2 + 6v_1vw^4 + v_1w^6)\\
&+ 96p_5(-6w_1uv - 12w_1uw^2 - 3u_1u^2 - 6u_1vw - 4u_1w^3 - 6v_1uw)\\
&+ 420p_6(-2w_1u^2 + 2w_1vw + w_1w^3 - 4u_1uw + 2v_1v + v_1w^2)\\
&+ 2240p_7(u_1v + v_1u)\\
&+ 7392p_9(w_1u + u_1w)\\
&- 2772p_{10}(w_1w + v_1)\\
&- 8640p_{11}u_1\\
&+ 4u(-81u^4 - 1728u^2vw - 768u^2w^3 + 768v^3 + 768v^2w^2 - 64vw^4 - 192w^6))/5,
\end{aligned}
$$

$$
\begin{aligned}
S^v(12,2) = {}&(5t(4w_1u^5 + 48w_1u^3vw + 72w_1u^3w^3 - 8w_1uv^3 - 84w_1uv^2w^2 - 78w_1uvw^4\\
&- 19w_1uw^6 + 28u_1u^4w - 24u_1u^2v^2 + 120u_1u^2vw^2 + 66u_1u^2w^4 - 24u_1v^3w\\
&- 36u_1v^2w^3 - 18u_1vw^5 - 3u_1w^7 - 16v_1u^3v + 40v_1u^3w^2 - 72v_1uv^2w - 72v_1uvw^3 - 18v_1uw^5)\\
&+ 96p_5(6w_1u^2w - 18w_1vw^2 - 18w_1w^4 - 6u_1uv + 12u_1uw^2 - 3v_1u^2 - 12v_1vw - 10v_1w^3)\\
&+ 420p_6(-2w_1uv - 7w_1uw^2 - 2u_1u^2 - 6u_1vw - 3u_1w^3 - 6v_1uw)\\
&+ 2240p_7(-w_1u^2 - 3w_1w^3 - 4u_1uw + v_1v - v_1w^2)\\
&+ 7392p_9(3w_1w^2 + u_1u + 2v_1w)\\
&+ 2772p_{10}(w_1u + 3u_1w)\\
&- 8640p_{11}v_1\\
&+ 8(-216u^4v + 567u^4w^2 - 1872u^2v^2w - 1520u^2vw^3 - 188u^2w^5 + 128v^4 - 576v^2w^4\\
&- 768vw^6 - 312w^8))/5
\end{aligned}
$$

2.3. **Generating Functions.** Solving equation (9), (10) which in our case is of the form

$$
\begin{aligned}
0 &= -\widetilde{D_t}(G_w) + w\widetilde{D_x}(G_u) - u\widetilde{D_x}(G_v) + 2u_1G_v\\
0 &= \widetilde{D_x}(G_w) - \widetilde{D_t}(G_u) - 3w\widetilde{D_x}(G_v) - 2w_1G_v
\end{aligned}
$$

$$0 = \widetilde{D_x}(G_u) - \widetilde{D_t}(G_v)$$

we found a number of solutions that generate infinite hierarchies and which are used to construct *nonlocal forms* (see Subsection 2.5 below). These generating functions are:

2.3.1. $(x, t)$ *independent Generating Functions.*

$$
\begin{aligned}
G_w(2) &= 1,\\
G_u(2) &= 0,\\
G_v(2) &= 0,\\
G_w(3) &= 0,\\
G_u(3) &= 1,\\
G_v(3) &= 0,\\
G_w(4) &= w,\\
G_u(4) &= 0,\\
G_v(4) &= 1/2,
\end{aligned}
$$

$$
\begin{aligned}
G_w(6) &= (v + 3w^2),\\
G_u(6) &= u,\\
G_v(6) &= w,\\
G_w(7) &= uw,\\
G_u(7) &= (2v + w^2)/2,\\
G_v(7) &= u,\\
G_w(8) &= (-3u^2 + 12vw + 14w^3)/14,\\
G_u(8) &= (-3uw)/7,\\
G_v(8) &= (3(v + w^2))/7,
\end{aligned}
$$

$$
\begin{aligned}
G_w(10) &= (2v^2 + 12vw^2 + 11w^4)/11,\\
G_u(10) &= 4uv/11,\\
G_v(10) &= (2(u^2 + 2vw + 2w^3))/11,\\
G_w(11) &= u(-2u^2 + 6vw + 3w^3)/3,\\
G_u(11) &= (-8u^2w + 4v^2 + 4vw^2 + w^4)/4,\\
G_v(11) &= u(2v + w^2),\\
G_w(12) &= 2(-3u^2v - 6u^2w^2 + 6v^2w + 14vw^3 + 9w^5),\\
G_u(12) &= 2u(-u^2 - 6vw - 4w^3),\\
G_v(12) &= (-6u^2w + 6v^2 + 12vw^2 + 7w^4),
\end{aligned}
$$

$$
\begin{aligned}
G_w(14) &= (-15u^4 - 60u^2w^3 + 20v^3 + 180v^2w^2 + 330vw^4 + 182w^6)/182,\\
G_u(14) &= 15u(-2u^2w + 2v^2 - w^4)/91,\\
G_v(14) &= 3(10u^2v + 10v^2w + 20vw^3 + 11w^5)/91,\\
G_w(15) &= u(-8u^2v - 12u^2w^2 + 12v^2w + 12vw^3 + 3w^5)/3,
\end{aligned}
$$

$$
\begin{aligned}
G_u(15) &= (-4u^4 - 48u^2vw - 24u^2w^3 + 8v^3 + 12v^2w^2 + 6vw^4 + w^6)/6,\\
G_v(15) &= u(-8u^2w + 12v^2 + 12vw^2 + 3w^4)/3,\\
G_w(16) &= ((14u^4w - 42u^2v^2 - 168u^2vw^2 - 105u^2w^4 + 56v^3w + 196v^2w^3 + 252vw^5 + 114w^7))/114,\\
G_u(16) &= (7u(-2u^2v + 2u^2w^2 - 6v^2w - 8vw^3 - 3w^5))/57,\\
G_v(16) &= (7(-u^4 - 12u^2vw - 8u^2w^3 + 4v^3 + 12v^2w^2 + 14vw^4 + 6w^6))/114
\end{aligned}
$$

2.3.2. *nonlocal Generating Functions.*

$$
\begin{aligned}
G_w(5) &= 0,\\
G_u(5) &= 0,\\
G_v(5) &= 0,
\end{aligned}
$$

$$
\begin{aligned}
G_w(9) &= (-42p_7 - 84p_5w + 20p_3(v + 3w^2) + 12p_2uw + p_1(-3u^2 + 12vw + 14w^3))/14,\\
G_u(9) &= (-14p_6 + 10p_3u + 3p_2(2v + w^2) - 3p_1uw)/7,\\
G_v(9) &= (-21p_5 + 10p_3w + 6p_2u + 3p_1(v + w^2))/7,
\end{aligned}
$$

$$
\begin{aligned}
G_w(13,1) &= (195p_{11} + 484p_9w + 126p_7(-v - 3w^2) - 70p_6uw + 14p_5(3u^2 - 12vw - 14w^3)\\
&\quad + 5p_3(2v^2 + 12vw^2 + 11w^4) + p_2u(-2u^2 + 6vw + 3w^3))/3,\\
G_u(13,1) &= (297p_{10} - 504p_7u + 140p_6(-2v - w^2) + 336p_5uw\\
&\quad + 80p_3uv + 3p_2(-8u^2w + 4v^2 + 4vw^2 + w^4))/12,\\
G_v(13,1) &= (242p_9 - 126p_7w - 70p_6u - 84p_5(v + w^2)\\
&\quad + 10p_3(u^2 + 2vw + 2w^3) + 3p_2u(2v + w^2))/3,\\
G_w(13,2) &= (-351p_{11} - 847p_9w + 210p_7(v + 3w^2) + 112p_6uw + 21p_5(-3u^2 + 12vw + 14w^3)\\
&\quad + 5p_3(-2v^2 - 12vw^2 - 11w^4) + p_1(-3u^2v - 6u^2w^2 + 6v^2w + 14vw^3 + 9w^5))/9,\\
G_u(13,2) &= (-132p_{10} + 210p_7u + 56p_6(2v + w^2) - 126p_5uw\\
&\quad - 20p_3uv + p_1u(-u^2 - 6vw - 4w^3))/9,\\
G_v(13,2) &= (-847p_9 + 420p_7w + 224p_6u + 252p_5(v + w^2)\\
&\quad + 20p_3(-u^2 - 2vw - 2w^3) + p_1(-6u^2w + 6v^2 + 12vw^2 + 7w^4))/18,\\
G_w(17,1) &= (10530p_{11}(v + 3w^2) + 3564p_{10}uw + 1694p_9(-3u^2 + 12vw + 14w^3)\\
&\quad + 945p_7(-2v^2 - 12vw^2 - 11w^4) + 280p_6u(2u^2 - 6vw - 3w^3)\\
&\quad + 252p_5(3u^2v + 6u^2w^2 - 6v^2w - 14vw^3 - 9w^5)\\
&\quad + p_3(-15u^4 - 60u^2w^3 + 20v^3 + 180v^2w^2 + 330vw^4 + 182w^6)\\
&\quad + 39(-924p_{13}w - 323p_{15}))/182,\\
G_u(17,1) &= (5265p_{11}u + 891p_{10}(2v + w^2) - 5082p_9uw\\
&\quad - 1890p_7uv + 105p_6(8u^2w - 4v^2 - 4vw^2 - w^4)\\
&\quad + 126p_5u(u^2 + 6vw + 4w^3)\\
&\quad + 15p_3u(-2u^2w + 2v^2 - w^4)\\
&\quad - 1170p_{14})/91,\\
G_v(17,1) &= (5265p_{11}w + 1782p_{10}u + 5082p_9(v + w^2)\\
&\quad + 945p_7(-u^2 - 2vw - 2w^3) + 420p_6u(-2v - w^2)
\end{aligned}
$$

$$
\begin{aligned}
&+ 63p_5(6u^2w - 6v^2 - 12vw^2 - 7w^4)\\
&+ 3p_3(10u^2v + 10v^2w + 20vw^3 + 11w^5)\\
&- 9009p_{13})/91,\\
G_w(17,2) &= (6240p_{11}(-v - 3w^2) - 2079p_{10}uw + 968p_9(3u^2 - 12vw - 14w^3)\\
&+ 504p_7(2v^2 + 12vw^2 + 11w^4) + 140p_6u(-2u^2 + 6vw + 3w^3)\\
&+ 112p_5(-3u^2v - 6u^2w^2 + 6v^2w + 14vw^3 + 9w^5)\\
&+ p_2u(-8u^2v - 12u^2w^2 + 12v^2w + 12vw^3 + 3w^5)\\
&+ 24(910p_{13}w + 323p_{15}))/3,\\
G_u(17,2) &= (-12480p_{11}u + 2079p_{10}(-2v - w^2) + 11616p_9uw\\
&+ 4032p_7uv + 210p_6(-8u^2w + 4v^2 + 4vw^2 + w^4)\\
&+ 224p_5u(-u^2 - 6vw - 4w^3)\\
&+ p_2(-4u^4 - 48u^2vw - 24u^2w^3 + 8v^3 + 12v^2w^2 + 6vw^4 + w^6)\\
&+ 2860p_{14})/6,\\
G_v(17,2) &= (-6240p_{11}w - 2079p_{10}u - 5808p_9(v + w^2)\\
&+ 1008p_7(u^2 + 2vw + 2w^3) + 420p_6u(2v + w^2) + 56p_5(-6u^2w + 6v^2 + 12vw^2 + 7w^4)\\
&+ p_2u(-8u^2w + 12v^2 + 12vw^2 + 3w^4)\\
&+ 10920p_{13})/3,\\
G_w(17,3) &= (39312p_{11}(v + 3w^2) + 12936p_{10}uw + 5929p_9(-3u^2 + 12vw + 14w^3)\\
&+ 2940p_7(-2v^2 - 12vw^2 - 11w^4) + 784p_6u(2u^2 - 6vw - 3w^3)\\
&+ 588p_5(3u^2v + 6u^2w^2 - 6v^2w - 14vw^3 - 9w^5)\\
&+ p_1(14u^4w - 42u^2v^2 - 168u^2vw^2 - 105u^2w^4 + 56v^3w + 196v^2w^3 + 252vw^5 + 114w^7)\\
&+ 52(-2695p_{13}w - 969p_{15}))/114,\\
G_u(17,3) &= (19656p_{11}u + 3234p_{10}(2v + w^2) - 17787p_9uw\\
&- 5880p_7uv + 294p_6(8u^2w - 4v^2 - 4vw^2 - w^4)\\
&+ 294p_5u(u^2 + 6vw + 4w^3)\\
&+ 7p_1u(-2u^2v + 2u^2w^2 - 6v^2w - 8vw^3 - 3w^5)\\
&- 4620p_{14})/57,\\
G_v(17,3) &= (39312p_{11}w + 12936p_{10}u + 35574p_9(v + w^2)\\
&+ 5880p_7(-u^2 - 2vw - 2w^3) + 2352p_6u(-2v - w^2)\\
&+ 294p_5(6u^2w - 6v^2 - 12vw^2 - 7w^4)\\
&+ 7p_1(-u^4 - 12u^2vw - 8u^2w^3 + 4v^3 + 12v^2w^2 + 14vw^4 + 6w^6)\\
&- 70070p_{13})/114,
\end{aligned}
$$

2.3.3. $(x,t)$ *dependent Generating Functions.*

$$
\begin{aligned}
G_w(1,1) &= x,\\
G_u(1,1) &= t,\\
G_v(1,1) &= 0,\\
G_w(5,1) &= x(v + 3w^2) - 6p_1w - 5p_3,\\
G_u(5,1) &= xu - 4p_2,
\end{aligned}
$$

$$
\begin{aligned}
G_v(5,1) &= xw - 3p_1,\\
G_w(5,2) &= tuw + 4p_1w + 4p_3,\\
G_u(5,2) &= (t(2v + w^2) + 6p_2)/2,\\
G_v(5,2) &= tu + 2p_1,\\
G_w(9,1) &= (x(2v^2 + 12vw^2 + 11w^4) + 8p_2uw + 20p_3(v + 3w^2) - 112p_5w - 63p_7)/11,\\
G_u(9,1) &= (4xuv + 4p_2(2v + w^2) + 20p_3u - 40p_6)/11,\\
G_v(9,1) &= (2x(u^2 + 2vw + 2w^3) + 8p_2u + 20p_3w - 56p_5)/11,\\
G_w(9,2) &= (tu(-2u^2 + 6vw + 3w^3) - 6p_2uw + 16p_3(-v - 3w^2) + 96p_5w + 56p_7)/3,\\
G_u(9,2) &= (3t(-8u^2w + 4v^2 + 4vw^2 + w^4) + 12p_2(-2v - w^2) - 64p_3u + 140p_6)/12,\\
G_v(9,2) &= (3tu(2v + w^2) - 6p_2u - 16p_3w + 48p_5)/3,\\
G_w(13,1) &= (x(-15u^4 - 60u^2w^3 + 20v^3 + 180v^2w^2 + 330vw^4 + 182w^6) + 50p_3(-2v^2 - 12vw^2 - 11w^4)\\
&\quad + 280p_5(-3u^2 + 12vw + 14w^3) + 1600p_6uw + 3150p_7(v + 3w^2) - 13552p_9w - 5850p_{11})/182,\\
G_u(13,1) &= (15xu(-2u^2w + 2v^2 - w^4) - 100p_3uv\\
&\quad - 840p_5uw + 400p_6(2v + w^2) + 1575p_7u - 1080p_{10})/91,\\
G_v(13,1) &= (3x(10u^2v + 10v^2w + 20vw^3 + 11w^5) + 50p_3(-u^2 - 2vw - 2w^3)\\
&\quad + 840p_5(v + w^2) + 800p_6u + 1575p_7w - 3388p_9)/91,\\
G_w(13,2) &= (tu(-8u^2v - 12u^2w^2 + 12v^2w + 12vw^3 + 3w^5) + 4p_3(2v^2 + 12vw^2 + 11w^4)\\
&\quad + 24p_5(3u^2 - 12vw - 14w^3) - 140p_6uw + 280p_7(-v - 3w^2) + 1232p_9w + 540p_{11})/3,\\
G_u(13,2) &= (t(-4u^4 - 48u^2vw - 24u^2w^3 + 8v^3 + 12v^2w^2 + 6vw^4 + w^6) + 32p_3uv\\
&\quad + 288p_5uw + 140p_6(-2v - w^2) - 560p_7u + 396p_{10})/6,\\
G_v(13,2) &= (tu(-8u^2w + 12v^2 + 12vw^2 + 3w^4) + 8p_3(u^2 + 2vw + 2w^3)\\
&\quad - 144p_5(v + w^2) - 140p_6u - 280p_7w + 616p_9)/3,
\end{aligned}
$$

2.4. **Nonlocal Vectors.** We consider now the $\ell^*$-extension of equation (1). The additional coordinates on this extension are denoted by $\Pi_w, \Pi_u, \Pi_v, \ldots, \Pi_{w_i} = \widetilde{D}_x^i(\Pi_w), \ldots$, and satisfy the condition

$$
\begin{aligned}
0 &= -\widetilde{D_t}(\Pi_w) + w\widetilde{D_x}(\Pi_u) - u\widetilde{D_x}(\Pi_v) + 2u_1\Pi_v\\
0 &= \widetilde{D_x}(\Pi_w) - \widetilde{D_t}(\Pi_u) - 3w\widetilde{D_x}(\Pi_v) - 2w_1\Pi_v\\
0 &= \widetilde{D_x}(\Pi_u) - \widetilde{D_t}(\Pi_v)
\end{aligned}
$$

We introduce nonlocal variables in the $\ell^*$-extension that we call *nonlocal vectors* and which are defined by

$$
\begin{aligned}
\widetilde{D}_x\Pi_{-4} &= S^w(-4)\Pi_w + S^u(-4)\Pi_u + S^v(-4)\Pi_v,\\
\widetilde{D}_t\Pi_{-4} &= (S^u(-4))\Pi_w + (S^v(-4) + S^w(-4)w)\Pi_u + (-3S^u(-4)w - S^w(-4)u)\Pi_v,\\
\widetilde{D}_x\Pi_1 &= S^w(1)\Pi_w + S^u(1)\Pi_u + S^v(1)\Pi_v,\\
\widetilde{D}_t\Pi_1 &= (S^u(1))\Pi_w + (S^v(1) + S^w(1)w)\Pi_u + (-3S^u(1)w - S^w(1)u)\Pi_v,\\
\widetilde{D}_x\Pi_2 &= S^w(2)\Pi_w + S^u(2)\Pi_u + S^v(2)\Pi_v,\\
\widetilde{D}_t\Pi_2 &= (S^u(2))\Pi_w + (S^v(2) + S^w(2)w)\Pi_u + (-3S^u(2)w - S^w(2)u)\Pi_v,\\
\widetilde{D}_x\Pi_3 &= S^w(3)\Pi_w + S^u(3)\Pi_u + S^v(3)\Pi_v,
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{D}_t\Pi_3 &= (S^u(3))\Pi_w + (S^v(3) + S^w(3)w)\Pi_u + (-3S^u(3)w - S^w(3)u)\Pi_v,\\
\widetilde{D}_x\Pi_4 &= S^w(4)\Pi_w + S^u(4)\Pi_u + S^v(4)\Pi_v,\\
\widetilde{D}_t\Pi_4 &= (S^u(4))\Pi_w + (S^v(4) + S^w(4)w)\Pi_u + (-3S^u(4)w - S^w(4)u)\Pi_v,\\
\widetilde{D}_x\Pi_5 &= S^w(5)\Pi_w + S^u(5)\Pi_u + S^v(5)\Pi_v,\\
\widetilde{D}_t\Pi_5 &= (S^u(5))\Pi_w + (S^v(5) + S^w(5)w)\Pi_u + (-3S^u(5)w - S^w(5)u)\Pi_v,\\
\widetilde{D}_x\Pi_6 &= S^w(6)\Pi_w + S^u(6)\Pi_u + S^v(6)\Pi_v,\\
\widetilde{D}_t\Pi_6 &= (S^u(6))\Pi_w + (S^v(6) + S^w(6)w)\Pi_u + (-3S^u(6)w - S^w(6)u)\Pi_v,\\
\widetilde{D}_x\Pi_7 &= S^w(7)\Pi_w + S^u(7)\Pi_u + S^v(7)\Pi_v,\\
\widetilde{D}_t\Pi_7 &= (S^u(7))\Pi_w + (S^v(7) + S^w(7)w)\Pi_u + (-3S^u(7)w - S^w(7)u)\Pi_v,\\
\widetilde{D}_x\Pi_{8,1} &= S^w(8,1)\Pi_w + S^u(8,1)\Pi_u + S^v(8,1)\Pi_v,\\
\widetilde{D}_t\Pi_{8,1} &= (S^u(8,1))\Pi_w + (S^v(8,1) + S^w(8,1)w)\Pi_u + (-3S^u(8,1)w - S^w(8,1)u)\Pi_v,\\
\widetilde{D}_x\Pi_{8,2} &= S^w(8,2)\Pi_w + S^u(8,2)\Pi_u + S^v(8,2)\Pi_v,\\
\widetilde{D}_t\Pi_{8,2} &= (S^u(8,2))\Pi_w + (S^v(8,2) + S^w(8,2)w)\Pi_u + (-3S^u(8,2)w - S^w(8,2)u)\Pi_v,\\
\widetilde{D}_x\Pi_9 &= S^w(9)\Pi_w + S^u(9)\Pi_u + S^v(9)\Pi_v,\\
\widetilde{D}_t\Pi_9 &= (S^u(9))\Pi_w + (S^v(9) + S^w(9)w)\Pi_u + (-3S^u(9)w - S^w(9)u)\Pi_v,\\
\widetilde{D}_x\Pi_{10} &= S^w(10)\Pi_w + S^u(10)\Pi_u + S^v(10)\Pi_v,\\
\widetilde{D}_t\Pi_{10} &= (S^u(10))\Pi_w + (S^v(10) + S^w(10)w)\Pi_u + (-3S^u(10)w - S^w(10)u)\Pi_v,\\
\widetilde{D}_x\Pi_{11} &= S^w(11)\Pi_w + S^u(11)\Pi_u + S^v(11)\Pi_v,\\
\widetilde{D}_t\Pi_{11} &= (S^u(11))\Pi_w + (S^v(11) + S^w(11)w)\Pi_u + (-3S^u(11)w - S^w(11)u)\Pi_v,\\
\widetilde{D}_x\Pi_{12,1} &= S^w(12,1)\Pi_w + S^u(12,1)\Pi_u + S^v(12,1)\Pi_v,\\
\widetilde{D}_t\Pi_{12,1} &= (S^u(12,1))\Pi_w + (S^v(12,1) + S^w(12,1)w)\Pi_u + (-3S^u(12,1)w - S^w(12,1)u)\Pi_v,\\
\widetilde{D}_x\Pi_{12,2} &= S^w(12,2)\Pi_w + S^u(12,2)\Pi_u + S^v(12,2)\Pi_v,\\
\widetilde{D}_t\Pi_{12,2} &= (S^u(12,2))\Pi_w + (S^v(12,2) + S^w(12,2)w)\Pi_u + (-3S^u(12,2)w - S^w(12,2)u)\Pi_v,\\
\widetilde{D}_x\Pi_{12,3} &= S^w(12,3)\Pi_w + S^u(12,3)\Pi_u + S^v(12,3)\Pi_v,\\
\widetilde{D}_t\Pi_{12,3} &= (S^u(12,3))\Pi_w + (S^v(12,3) + S^w(12,3)w)\Pi_u + (-3S^u(12,3)w - S^w(12,3)u)\Pi_v,\\
\widetilde{D}_x\Pi_{13} &= S^w(13)\Pi_w + S^u(13)\Pi_u + S^v(13)\Pi_v,\\
\widetilde{D}_t\Pi_{13} &= (S^u(13))\Pi_w + (S^v(13) + S^w(13)w)\Pi_u + (-3S^u(13)w - S^w(13)u)\Pi_v,\\
\widetilde{D}_x\Pi_{14} &= S^w(14)\Pi_w + S^u(14)\Pi_u + S^v(14)\Pi_v,\\
\widetilde{D}_t\Pi_{14} &= (S^u(14))\Pi_w + (S^v(14) + S^w(14)w)\Pi_u + (-3S^u(14)w - S^w(14)u)\Pi_v,\\
\widetilde{D}_x\Pi_{15} &= S^w(15)\Pi_w + S^u(15)\Pi_u + S^v(15)\Pi_v,\\
\widetilde{D}_t\Pi_{15} &= (S^u(15))\Pi_w + (S^v(15) + S^w(15)w)\Pi_u + (-3S^u(15)w - S^w(15)u)\Pi_v,
\end{aligned}
$$

where the symmetries $S(i); i = \dots$ have been described in Subsection 2.2.
The name arises from the fact that the densities of these conservation laws, which in this case read

$$S(*)^w\Pi_w + S(*)^u\Pi_u + S(*)^v\Pi_v,$$

are just formally equivalent to the Symmetries (vector fields)

$$S(*)^w\partial_w + S(*)^u\partial_u + S(*)^v\partial_v.$$

Formally $\Pi(*)$ are denoted by

$$\Pi(*) = D_x^{-1}(S(*)^w\Pi_w + S(*)^u\Pi_u + S(*)^v\Pi_v),$$

compatibility conditions being satisfied.

2.5. **Nonlocal Forms.** Passing to the $\ell$-extension of equation (1), we introduce the additional coordinates on this extension that are denoted by $\Omega^w, \Omega^u, \Omega^v, \ldots, \Omega^{w_i} = \widetilde{D}_x^i(\Omega^w), \ldots$, satisfying the condition

$$\begin{aligned}
0 &= -\widetilde{D}_t(\Omega^w) + \widetilde{D}_x(\Omega^u),\\
0 &= -\widetilde{D}_t(\Omega^u) + w\widetilde{D}_x(\Omega^w) + w_1\Omega^w + \widetilde{D}_x(\Omega^v),\\
0 &= -\widetilde{D}_t(\Omega^v) - u\widetilde{D}_x(\Omega^w) - 3u_1\Omega^w - 3w\widetilde{D}_x(\Omega^u) - w_1\Omega^u
\end{aligned}$$

We introduce nonlocal variables in the $\ell$-extension called *nonlocal forms* and described by

$$\begin{aligned}
\widetilde{D}_x\Omega(2) &= G_w(2)\Omega^w + G_u(2)\Omega^u + G_v(2)\Omega^v,\\
\widetilde{D}_t\Omega(2) &= (G_u(2)w - G_v(2)u)\Omega^w + (-3G_v(2)w + G_w(2))\Omega^u + G_u(2)\Omega^v,\\
\widetilde{D}_x\Omega(3) &= G_w(3)\Omega^w + G_u(3)\Omega^u + G_v(3)\Omega^v,\\
\widetilde{D}_t\Omega(3) &= (G_u(3)w - G_v(3)u)\Omega^w + (-3G_v(3)w + G_w(3))\Omega^u + G_u(3)\Omega^v,\\
\widetilde{D}_x\Omega(4) &= G_w(4)\Omega^w + G_u(4)\Omega^u + G_v(4)\Omega^v,\\
\widetilde{D}_t\Omega(4) &= (G_u(4)w - G_v(4)u)\Omega^w + (-3G_v(4)w + G_w(4))\Omega^u + G_u(4)\Omega^v,\\
\widetilde{D}_x\Omega(5) &= G_w(5)\Omega^w + G_u(5)\Omega^u + G_v(5)\Omega^v,\\
\widetilde{D}_t\Omega(5) &= (G_u(5)w - G_v(5)u)\Omega^w + (-3G_v(5)w + G_w(5))\Omega^u + G_u(5)\Omega^v,\\
\widetilde{D}_x\Omega(6) &= G_w(6)\Omega^w + G_u(6)\Omega^u + G_v(6)\Omega^v,\\
\widetilde{D}_t\Omega(6) &= (G_u(6)w - G_v(6)u)\Omega^w + (-3G_v(6)w + G_w(6))\Omega^u + G_u(6)\Omega^v,\\
\widetilde{D}_x\Omega(7) &= G_w(7)\Omega^w + G_u(7)\Omega^u + G_v(7)\Omega^v,\\
\widetilde{D}_t\Omega(7) &= (G_u(7)w - G_v(7)u)\Omega^w + (-3G_v(7)w + G_w(7))\Omega^u + G_u(7)\Omega^v,\\
\widetilde{D}_x\Omega(8) &= G_w(8)\Omega^w + G_u(8)\Omega^u + G_v(8)\Omega^v,\\
\widetilde{D}_t\Omega(8) &= (G_u(8)w - G_v(8)u)\Omega^w + (-3G_v(8)w + G_w(8))\Omega^u + G_u(8)\Omega^v,\\
\widetilde{D}_x\Omega(9) &= G_w(9)\Omega^w + G_u(9)\Omega^u + G_v(9)\Omega^v,\\
\widetilde{D}_t\Omega(9) &= (G_u(9)w - G_v(9)u)\Omega^w + (-3G_v(9)w + G_w(9))\Omega^u + G_u(9)\Omega^v,\\
\widetilde{D}_x\Omega(10) &= G_w(10)\Omega^w + G_u(10)\Omega^u + G_v(10)\Omega^v,\\
\widetilde{D}_t\Omega(10) &= (G_u(10)w - G_v(10)u)\Omega^w + (-3G_v(10)w + G_w(10))\Omega^u + G_u(10)\Omega^v,\\
\widetilde{D}_x\Omega(11) &= G_w(11)\Omega^w + G_u(11)\Omega^u + G_v(11)\Omega^v,\\
\widetilde{D}_t\Omega(11) &= (G_u(11)w - G_v(11)u)\Omega^w + (-3G_v(11)w + G_w(11))\Omega^u + G_u(11)\Omega^v,\\
\widetilde{D}_x\Omega(12) &= G_w(12)\Omega^w + G_u(12)\Omega^u + G_v(12)\Omega^v,\\
\widetilde{D}_t\Omega(12) &= (G_u(12)w - G_v(12)u)\Omega^w + (-3G_v(12)w + G_w(12))\Omega^u + G_u(12)\Omega^v,\\
\widetilde{D}_x\Omega(13,1) &= G_w(13,1)\Omega^w + G_u(13,1)\Omega^u + G_v(13,1)\Omega^v,\\
\widetilde{D}_t\Omega(13,1) &= (G_u(13,1)w - G_v(13,1)u)\Omega^w + (-3G_v(13,1)w + G_w(13,1))\Omega^u + G_u(13,1)\Omega^v,
\end{aligned}$$

$$\widetilde{D}_x\Omega(13,2) = G_w(13,2)\Omega^w + G_u(13,2)\Omega^u + G_v(13,2)\Omega^v,$$
$$\widetilde{D}_t\Omega(13,2) = (G_u(13,2)w - G_v(13,2)u)\Omega^w + (-3G_v(13,2)w + G_w(13,2))\Omega^u + G_u(13,2)\Omega^v,$$
$$\widetilde{D}_x\Omega(14) = G_w(14)\Omega^w + G_u(14)\Omega^u + G_v(14)\Omega^v,$$
$$\widetilde{D}_t\Omega(14) = (G_u(14)w - G_v(14)u)\Omega^w + (-3G_v(14)w + G_w(14))\Omega^u + G_u(14)\Omega^v,$$
$$\widetilde{D}_x\Omega(15) = G_w(15)\Omega^w + G_u(15)\Omega^u + G_v(15)\Omega^v,$$
$$\widetilde{D}_t\Omega(15) = (G_u(15)w - G_v(15)u)\Omega^w + (-3G_v(15)w + G_w(15))\Omega^u + G_u(15)\Omega^v,$$
$$\widetilde{D}_x\Omega(16) = G_w(16)\Omega^w + G_u(16)\Omega^u + G_v(16)\Omega^v,$$
$$\widetilde{D}_t\Omega(16) = (G_u(16)w - G_v(16)u)\Omega^w + (-3G_v(16)w + G_w(16))\Omega^u + G_u(16)\Omega^v,$$
$$\widetilde{D}_x\Omega(17,1) = G_w(17,1)\Omega^w + G_u(17,1)\Omega^u + G_v(17,1)\Omega^v,$$
$$\widetilde{D}_t\Omega(17,1) = (G_u(17,1)w - G_v(17,1)u)\Omega^w + (-3G_v(17,1)w + G_w(17,1))\Omega^u + G_u(17,1)\Omega^v,$$
$$\widetilde{D}_x\Omega(17,2) = G_w(17,2)\Omega^w + G_u(17,2)\Omega^u + G_v(17,2)\Omega^v,$$
$$\widetilde{D}_t\Omega(17,2) = (G_u(17,2)w - G_v(17,2)u)\Omega^w + (-3G_v(17,2)w + G_w(17,2))\Omega^u + G_u(17,2)\Omega^v,$$
$$\widetilde{D}_x\Omega(17,3) = G_w(17,3)\Omega^w + G_u(17,3)\Omega^u + G_v(17,3)\Omega^v,$$
$$\widetilde{D}_t\Omega(17,3) = (G_u(17,3)w - G_v(17,3)u)\Omega^w + (-3G_v(17,3)w + G_w(17,3))\Omega^u + G_u(17,3)\Omega^v,$$

where the generating functions $G(i), \dots$ have been described in Subsection 2.3. The name arises from the fact that the densities of these conservation laws, which in this case read

$$G(*)_w\Omega^w + G(*)_u\Omega^u + G(*)_v\Omega^v,$$

are just formally equivalent to the Generating Form

$$G(*)_w d^w + G(*)_u d^u + G(*)_v d^v.$$

Formally $\Omega(*)$ are denoted by

$$\Omega(*) = D_x^{-1}(G(*)_w\Omega^w + G(*)_u\Omega^u + G(*)_v\Omega^v),$$

compatibility conditions being satisfied.

2.6. **Recursion Operators for Symmetries.** Using the method described in Subsection 1.5, we found nontrivial solutions of the linearized equation in the $\ell$-extension enriched with nonlocal variables. The first few of them are

$$R_S^w(0) = \Omega^w,$$
$$R_S^u(0) = \Omega^u,$$
$$R_S^v(0) = \Omega^v,$$

$$R_S^w(4,1) = (2\Omega(4)w_1 + \Omega(3)u_1 + \Omega(2)(2w_1w + v_1))/2,$$
$$R_S^u(4,1) = (2\Omega(4)u_1 + \Omega(3)(w_1w + v_1) - \Omega(2)(w_1u + u_1w))/2,$$
$$R_S^v(4,1) = (2\Omega(4)v_1 + \Omega(3)(-w_1u - 3u_1w) + \Omega(2)(-3w_1w^2 - u_1u - 2v_1w))/2,$$

$$R_S^w(4,2) = (-4\Omega(4)w_1 - \Omega(3)u_1 + 2\Omega^v w + 3\Omega^u u + 4\Omega^w(v + 2w^2))/8,$$
$$R_S^u(4,2) = (-4\Omega(4)u_1 - \Omega(3)(w_1w + v_1) + 3\Omega^v u + 2\Omega^u(2v + w^2) + \Omega^w uw)/8,$$
$$R_S^v(4,2) = (-4\Omega(4)v_1 + \Omega(3)(w_1u + 3u_1w) + 4\Omega^v v - 11\Omega^u uw + 3\Omega^w(-u^2 - 2w^3))/8,$$

$$R_S^w(8,1) = (-28\Omega(8)w_1 - 10\Omega(7)u_1 + 8\Omega(6)(-2w_1w - v_1)$$

$$
\begin{aligned}
&+ 8\Omega(4)(w_1v + 3w_1w^2 + u_1u + v_1w) + \Omega(3)(2w_1uw + 2u_1v + u_1w^2 + 2v_1u))/2,\\
R^u_S(8,1) &= (-28\Omega(8)u_1 - 10\Omega(7)(w_1w + v_1) + 8\Omega(6)(w_1u + u_1w)\\
&+ 8\Omega(4)(u_1v + v_1u) + \Omega(3)(-2w_1u^2 + 2w_1vw + w_1w^3 - 4u_1uw + 2v_1v + v_1w^2))/2,\\
R^v_S(8,1) &= (-28\Omega(8)v_1 + 10\Omega(7)(w_1u + 3u_1w) + 8\Omega(6)(3w_1w^2 + u_1u + 2v_1w)\\
&+ 8\Omega(4)(-w_1u^2 - 3w_1w^3 - 4u_1uw + v_1v - v_1w^2)\\
&+ \Omega(3)(-2w_1uv - 7w_1uw^2 - 2u_1u^2 - 6u_1vw - 3u_1w^3 - 6v_1uw))/2,
\end{aligned}
$$

$$
\begin{aligned}
R^w_S(8,2) &= (70\Omega(8)w_1 + 24\Omega(7)u_1 + 18\Omega(6)(2w_1w + v_1)\\
&+ 12\Omega(4)(-w_1v - 3w_1w^2 - u_1u - v_1w)\\
&+ \Omega(2)(-3w_1u^2 + 12w_1vw + 14w_1w^3 - 6u_1uw + 6v_1v + 6v_1w^2))/14,\\
R^u_S(8,2) &= (70\Omega(8)u_1 + 24\Omega(7)(w_1w + v_1) - 18\Omega(6)(w_1u + u_1w)\\
&- 12\Omega(4)(u_1v + v_1u)\\
&+ \Omega(2)(-6w_1uv - 12w_1uw^2 - 3u_1u^2 - 6u_1vw - 4u_1w^3 - 6v_1uw))/14,\\
R^v_S(8,2) &= (70\Omega(8)v_1 + 24\Omega(7)(-w_1u - 3u_1w) + 18\Omega(6)(-3w_1w^2 - u_1u - 2v_1w)\\
&+ 12\Omega(4)(w_1u^2 + 3w_1w^3 + 4u_1uw - v_1v + v_1w^2)\\
&+ \Omega(2)(6w_1u^2w - 18w_1vw^2 - 18w_1w^4 - 6u_1uv + 12u_1uw^2 - 3v_1u^2 - 12v_1vw - 10v_1w^3))/14,
\end{aligned}
$$

$$
\begin{aligned}
R^w_S(8,3) &= (-70\Omega(8)w_1 - 20\Omega(7)u_1 + 12\Omega(6)(-2w_1w - v_1)\\
&+ 4\Omega(4)(w_1v + 3w_1w^2 + u_1u + v_1w)\\
&+ \Omega^v(9u^2 + 16vw + 16w^3)\\
&+ 8\Omega^u u(3v + w^2) + \Omega^w(-3u^2w + 16v^2 + 64vw^2 + 52w^4))/52,\\
R^u_S(8,3) &= (-70\Omega(8)u_1 - 20\Omega(7)(w_1w + v_1) + 12\Omega(6)(w_1u + u_1w)\\
&+ 4\Omega(4)(u_1v + v_1u) + 8\Omega^v u(3v + w^2)\\
&+ 2\Omega^u(-15u^2w + 8v^2 + 8vw^2 + 2w^4) + \Omega^w u(-9u^2 + 8vw - 8w^3))/52,\\
R^v_S(8,3) &= (-70\Omega(8)v_1 + 20\Omega(7)(w_1u + 3u_1w) + 12\Omega(6)(3w_1w^2 + u_1u + 2v_1w)\\
&+ 4\Omega(4)(-w_1u^2 - 3w_1w^3 - 4u_1uw + v_1v - v_1w^2)\\
&+ \Omega^v(-39u^2w + 16v^2 - 12w^4)\\
&+ \Omega^u u(-9u^2 - 88vw - 40w^3)\\
&+ \Omega^w(-24u^2v - 35u^2w^2 - 48vw^3 - 48w^5))/52
\end{aligned}
$$

The operator corresponding to the first solution is just the identity operator.
We shall not write down here explicitly the classical presentation of the associated matrix operators .

2.7. **Recursion Operators for Generating Functions.** Using the method described in Subsection 1.6, we found nontrivial solutions of the adjoint linearized equation in the $\ell^*$-extension enriched with nonlocal variables. The first few of them are

$$
\begin{aligned}
R^w_G(0,1) &= \Pi_w,\\
R^u_G(0,1) &= \Pi_u,\\
R^v_G(0,1) &= \Pi_v,
\end{aligned}
$$

$$R_G^w(4,1) = (\Pi_3 + 2\Pi_1 w)/2,$$
$$R_G^u(4,1) = \Pi_2/2,$$
$$R_G^v(4,1) = \Pi_1/2,$$

$$R_G^w(4,2) = (-2\Pi_3 + 3\Pi_v(-2w^3 - u^2) + \Pi_u wu + 4\Pi_w(2w^2 + v))/8,$$
$$R_G^u(4,2) = (-\Pi_2 - 11\Pi_v wu + 2\Pi_u(w^2 + 2v) + 3\Pi_w u)/8,$$
$$R_G^v(4,2) = (4\Pi_v v + 3\Pi_u u + 2\Pi_w w)/8,$$

$$R_G^w(8,1) = -\Pi_7 - 8\Pi_5 w + 2\Pi_3(3w^2 + v) + \Pi_2 wu,$$
$$R_G^u(8,1) = (-5\Pi_6 + 4\Pi_3 u + \Pi_2(w^2 + 2v))/2,$$
$$R_G^v(8,1) = -4\Pi_5 + 2\Pi_3 w + \Pi_2 u,$$

$$R_G^w(8,2) = (5\Pi_7 + 36\Pi_5 w + 6\Pi_3(-3w^2 - v) + \Pi_1(14w^3 + 12wv - 3u^2))/14,$$
$$R_G^u(8,2) = (6\Pi_6 - 3\Pi_3 u - 3\Pi_1 wu)/7,$$
$$R_G^v(8,2) = (9\Pi_5 - 3\Pi_3 w + 3\Pi_1(w^2 + v))/7,$$

$$\begin{aligned} R_G^w(8,3) &= (-5\Pi_7 - 24\Pi_5 w + 2\Pi_3(3w^2 + v) + \Pi_v(-48w^5 - 48w^3 v - 35w^2u^2 - 24u^2 v) \\ &\quad + \Pi_u u(-8w^3 + 8wv - 9u^2) + \Pi_w(52w^4 + 64w^2 v - 3wu^2 + 16v^2))/52, \\ R_G^u(8,3) &= (-10\Pi_6 + 2\Pi_3 u + \Pi_v u(-40w^3 - 88wv - 9u^2) \\ &\quad + 2\Pi_u(2w^4 + 8w^2 v - 15wu^2 + 8v^2) + 8\Pi_w u(w^2 + 3v))/52, \\ R_G^v(8,3) &= (-12\Pi_5 + 2\Pi_3 w + \Pi_v(-12w^4 - 39wu^2 + 16v^2) \\ &\quad + 8\Pi_u u(w^2 + 3v) + \Pi_w(16w^3 + 16wv + 9u^2))/52 \end{aligned}$$

The operator corresponding to the first solution is just the identity operator.
We shall not write down here explicitly the classical presentation of the associated matrix operators .

2.8. **Hamiltonian and Noether Structures.** Using the method described in Subsection 1.7, we found three nontrivial solutions of the linearized equation in the $\ell^*$-extension enriched with nonlocal variables. The first one is

$$Ha^w(0,1) = \Pi_{v_1},$$
$$Ha^u(0,1) = \Pi_{u_1},$$
$$Ha^v(0,1) = -4\Pi_{v_1} w + \Pi_{w_1} - 2\Pi_v w_1,$$

$$\begin{aligned} Ha^w(4,1) &= (4\Pi_{v_1} v + 3\Pi_{u_1} u + 2\Pi_{w_1} w + 2\Pi_v v_1 + \Pi_u u_1)/2, \\ Ha^u(4,1) &= (-11\Pi_{v_1} wu + 2\Pi_{u_1}(w^2 + 2v) + 3\Pi_{w_1} u + 2\Pi_v(-wu_1 - 4uw_1) \\ &\quad + \Pi_u(ww_1 + v_1))/2, \\ Ha^v(4,1) &= (\Pi_{v_1}(-6w^3 - 16wv - 3u^2) - 11\Pi_{u_1} wu + 4\Pi_{w_1} v \\ &\quad + 2\Pi_v(-3w^2 w_1 - 2wv_1 - uu_1 - 4vw_1) \\ &\quad + \Pi_u(-3wu_1 - uw_1))/2, \end{aligned}$$

$$Ha^w(4,2) = \Pi_v v_1 + \Pi_u u_1 + \Pi_w w_1,$$

$$Ha^u(4,2) = \Pi_v(-3wu_1 - uw_1) + \Pi_u(ww_1 + v_1) + \Pi_w u_1,$$
$$Ha^v(4,2) = \Pi_v(-3w^2w_1 - 4wv_1 - uu_1) + \Pi_u(-3wu_1 - uw_1) + \Pi_w v_1,$$

$$Ha^w(8,1) = (\Pi_3 w_1 + \Pi_2 u_1 + \Pi_1(2ww_1 + v_1))/2,$$
$$Ha^u(8,1) = (\Pi_3 u_1 + \Pi_2(ww_1 + v_1) - \Pi_1(wu_1 + uw_1))/2,$$
$$Ha^v(8,1) = (\Pi_3 v_1 + \Pi_2(-3wu_1 - uw_1) + \Pi_1(-3w^2w_1 - 2wv_1 - uu_1))/2,$$

$$\begin{aligned}
Ha^w(8,2) = (&-6\Pi_3 w_1 - 2\Pi_2 u_1 \\
&+ \Pi_{v_1}(12w^4 + 39wu^2 - 16v^2) \\
&+ 8\Pi_{u_1} u(-w^2 - 3v) \\
&+ \Pi_{w_1}(-16w^3 - 16wv - 9u^2) \\
&+ \Pi_v(6w^3w_1 - 8w^2v_1 + 10wuu_1 + 21u^2w_1 - 12vv_1) \\
&+ 2\Pi_u(-w^2u_1 - 6wuw_1 - 3uv_1 - 2vu_1) \\
&+ 2\Pi_w(-wv_1 + 2vw_1))/4,
\end{aligned}$$
$$\begin{aligned}
Ha^u(8,2) = (&-6\Pi_3 u_1 - 2\Pi_2(ww_1 + v_1) \\
&+ \Pi_{v_1} u(40w^3 + 88wv + 9u^2) \\
&+ 2\Pi_{u_1}(-2w^4 - 8w^2v + 15wu^2 - 8v^2) \\
&+ 8\Pi_{w_1} u(-w^2 - 3v) \\
&+ \Pi_v(-2w^3u_1 + 34w^2uw_1 + 10wuv_1 + 4wvu_1 + 3u^2u_1 + 60uvw_1) \\
&+ 2\Pi_u(-w^3w_1 - w^2v_1 + 3wuu_1 - 2wvw_1 + 3u^2w_1 - 2vv_1) \\
&+ 2\Pi_w(3w^2u_1 + wuw_1 + 2vu_1))/4,
\end{aligned}$$
$$\begin{aligned}
Ha^v(8,2) = (&-6\Pi_3 v_1 + 2\Pi_2(3wu_1 + uw_1) \\
&+ \Pi_{v_1}(48w^3v - 121w^2u^2 + 64wv^2 + 24u^2v) \\
&+ \Pi_{u_1} u(40w^3 + 88wv + 9u^2) \\
&+ \Pi_{w_1}(12w^4 + 39wu^2 - 16v^2) \\
&+ \Pi_v(6w^3v_1 - 22w^2uu_1 + 36w^2vw_1 - 88wu^2w_1 + 16wvv_1 + 3u^2v_1 + 12uvu_1 + 32v^2w_1) \\
&+ 2\Pi_u(3w^3u_1 + 10w^2uw_1 + 6wuv_1 + 6wvu_1 + 3u^2u_1 + 2uvw_1) \\
&+ 2\Pi_w(3w^3w_1 + 4w^2v_1 + wuu_1 + 2vv_1))/4,
\end{aligned}$$

$$\begin{aligned}
Ha^w(8,3) = (&4\Pi_3 w_1 + \Pi_2 u_1 \\
&+ \Pi_{v_1}(-12w^4 - 39wu^2 + 16v^2) \\
&+ 8\Pi_{u_1} u(w^2 + 3v) \\
&+ \Pi_{w_1}(16w^3 + 16wv + 9u^2) \\
&+ \Pi_v(-12w^3w_1 + 8w^2v_1 - 21wuu_1 - 24u^2w_1 + 16vv_1) \\
&+ \Pi_u(4w^2u_1 + 13wuw_1 + 9uv_1 + 8vu_1) \\
&+ \Pi_w(8w^2w_1 + 4wv_1 + 3uu_1))/8,
\end{aligned}$$
$$\begin{aligned}
Ha^u(8,3) = (&4\Pi_3 u_1 + \Pi_2(ww_1 + v_1) \\
&+ \Pi_{v_1} u(-40w^3 - 88wv - 9u^2) \\
&+ 2\Pi_{u_1}(2w^4 + 8w^2v - 15wu^2 + 8v^2)
\end{aligned}$$

$$
\begin{aligned}
&+8\Pi_{w_1}u(w^2+3v)\\
&+\Pi_v(-4w^3u_1-45w^2uw_1-21wuv_1-16wvu_1-6u^2u_1-64uvw_1)\\
&+\Pi_u(4w^3w_1+4w^2v_1-14wuu_1+8wvw_1-9u^2w_1+8vv_1)\\
&+\Pi_w(-4w^2u_1-wuw_1+3uv_1))/8,\\
Ha^v(8,3)=&\,(4\Pi_3v_1+\Pi_2(-3wu_1-uw_1)\\
&+\Pi_{v_1}(-48w^3v+121w^2u^2-64wv^2-24u^2v)\\
&+\Pi_{u_1}u(-40w^3-88wv-9u^2)\\
&+\Pi_{w_1}(-12w^4-39wu^2+16v^2)\\
&+\Pi_v(-12w^3v_1+55w^2uu_1-48w^2vw_1+99wu^2w_1-32wvv_1-6u^2v_1-16uvu_1-32v^2w_1)\\
&+\Pi_u(-12w^3u_1-31w^2uw_1-23wuv_1-24wvu_1-9u^2u_1-8uvw_1)\\
&+\Pi_w(-12w^3w_1-8w^2v_1-13wuu_1-3u^2w_1))/8
\end{aligned}
$$

The operator corresponding to the first solution is

$$
H^1=\begin{bmatrix}0&0&D_x\\0&D_x&0\\D_x&0&-4wD_x-2w_1\end{bmatrix},
$$

being the first Hamiltonian operator, and

$$
H^{2,1}=\begin{bmatrix}wD_x&\frac{3}{2}uD_x+\frac{1}{2}u_1&2vD_x+v_1\\ \frac{3}{2}uD_x&(w^2+2v)D_x+\frac{1}{2}(ww_1+v_1)&-\frac{11}{2}wuD_x+(-wu_1-4uw_1)\\ 2vD_x&-\frac{11}{2}wuD_x+\frac{1}{2}(-3wu_1-uw_1)&B_{(2,2,1)}D_x+B_{(2,2,2)}\end{bmatrix}
$$

where

$$
B_{(2,2,1)}=-(6w^3+16wv+3u^2)/2,
$$

$$
B_{(2,2,2)}=-(3w^2w_1+2wv_1+uu_1+4vw_1),
$$

and

$$
H^{2,2}=\begin{bmatrix}w_1&u_1&v_1\\u_1&ww_1+v_1&-3wu_1-uw_1\\v_1&-3wu_1-uw_1&-3w^2w_1-4wv_1-uu_1\end{bmatrix}.
$$

Although $H^{2,1}$ and $H^{2,2}$ are not anti symmetric the combination

$$
H^{2,1}+(1/2)H^{2,2}
$$

is, and represents the second Hamiltonian operator, satisfying criteria (16) and (17)
.
So we here obtained the first examples of Noether operators $H^{2,1}$ and $H^{2,2}$ being not Hamiltonian.

2.9. **Symplectic and Inverse Noether Structures.** Using the method described in Subsection 1.8, we found nontrivial solutions of the adjoint linearized equation in the $\ell$-extension enriched with nonlocal variables. The first few of them are

$$
\begin{aligned}
Sy^w(-1) &= \Omega(4) + \Omega(2)w,\\
Sy^u(-1) &= \Omega(3)/2,\\
Sy^v(-1) &= \Omega(2)/2,\\
\\
Sy^w(3,1) &= -14\Omega(8) - 8\Omega(6)w + 4\Omega(4)(3w^2 + v) + \Omega(3)wu,\\
Sy^u(3,1) &= (-10\Omega(7) + 8\Omega(4)u + \Omega(3)(w^2 + 2v))/2,\\
Sy^v(3,1) &= -4\Omega(6) + 4\Omega(4)w + \Omega(3)u,\\
\\
Sy^w(3,2) &= (70\Omega(8) + 36\Omega(6)w + 12\Omega(4)(-3w^2 - v) + \Omega(2)(14w^3 + 12wv - 3u^2))/14,\\
Sy^u(3,2) &= (12\Omega(7) - 6\Omega(4)u - 3\Omega(2)wu)/7,\\
Sy^v(3,2) &= (9\Omega(6) - 6\Omega(4)w + 3\Omega(2)(w^2 + v))/7,\\
\\
Sy^w(7,1) &= (15\Omega(12) + 616\Omega(10)w + 140\Omega(8)(-3w^2 - v) - 40\Omega(7)wu\\
&\quad + 4\Omega(6)(-14w^3 - 12wv + 3u^2) + 2\Omega(4)(11w^4 + 12w^2v + 2v^2))/22,\\
Sy^u(7,1) &= (36\Omega(11) - 70\Omega(8)u + 10\Omega(7)(-w^2 - 2v)\\
&\quad + 12\Omega(6)wu + 4\Omega(4)uv)/11,\\
Sy^v(7,1) &= (154\Omega(10) - 70\Omega(8)w - 20\Omega(7)u\\
&\quad - 12\Omega(6)(w^2 + v) + 2\Omega(4)(2w^3 + 2wv + u^2))/11,\\
\\
Sy^w(7,2) &= (-40\Omega(12) - 1584\Omega(10)w + 336\Omega(8)(3w^2 + v) + 90\Omega(7)wu\\
&\quad + 8\Omega(6)(14w^3 + 12wv - 3u^2) + \Omega(3)u(3w^3 + 6wv - 2u^2))/3,\\
Sy^u(7,2) &= (-252\Omega(11) + 448\Omega(8)u + 60\Omega(7)(w^2 + 2v)\\
&\quad - 64\Omega(6)wu + \Omega(3)(w^4 + 4w^2v - 8wu^2 + 4v^2))/4,\\
Sy^v(7,2) &= -264\Omega(10) + 112\Omega(8)w + 30\Omega(7)u\\
&\quad + 16\Omega(6)(w^2 + v) + \Omega(3)u(w^2 + 2v),\\
\\
Sy^w(7,3) &= (18\Omega(12) + 693\Omega(10)w + 140\Omega(8)(-3w^2 - v) - 36\Omega(7)wu\\
&\quad + 3\Omega(6)(-14w^3 - 12wv + 3u^2) + \Omega(2)(9w^5 + 14w^3v - 6w^2u^2 + 6wv^2 - 3u^2v))/9,\\
Sy^u(7,3) &= (84\Omega(11) - 140\Omega(8)u + 18\Omega(7)(-w^2 - 2v)\\
&\quad + 18\Omega(6)wu + \Omega(2)u(-4w^3 - 6wv - u^2))/9,\\
Sy^v(7,3) &= (693\Omega(10) - 280\Omega(8)w - 72\Omega(7)u\\
&\quad - 36\Omega(6)(w^2 + v) + \Omega(2)(7w^4 + 12w^2v - 6wu^2 + 6v^2))/18
\end{aligned}
$$

A similar construction as presented for the Hamiltonian operators in the previous section applies for the construction of the Symplectic operators and the Inverse Noether operators.

## 3. Conclusion

In particular, we found recursion operators for symmetries and generating functions, Hamiltonian and Symplectic structures. Moreover we constructed Noether

and Inverse Noether operators, leading to multiple infinite hierarchies of all structures.
The research was based on new geometricalmethods giving rise to efficient computational algorithms.

Our experience shows that the methods applied are of a universal nature and may be used to analyze a lot of other equations, both classical and supersymmetric. In particular, from technical point of view, the canonical representation of nonlocal operators (see Subsection 1.9) seems to be quite efficient and convenient when dealing with such operators. Note that all nonlocal operators constructed in this paper are represented in the canonical form.

## Acknowledgments

I.K. and A.V. are grateful to the University of Twente, where this research was done, for hospitality. The work of A.V. was supported in part by the NWO and FOM (The Netherlands). We are also want to thank our anonymous referee for his/her extremely valuable comments.

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